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THE RESPONSE OF SIMPLE STRUCTURES TO DYNAMIC LOADS

By
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UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

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THE RESPONSE OF SIMPLE STRUCTURES TO DYNAMIC LOADS

I. INTRODUCTION

A. INTRODUCTORY STATEMENT

The response of an engineering structure to a dynamic disturbance depends upon the characteristics of the impulsive excitation and upon the physical properties of the structure itself. If the accuracy of the results is to be in keeping with the accuracy of the analytical procedure, the use of rigorous methods of analysis in calculating responses requires precise data describing the load and the structure involved. However, there are always certain inaccuracies present in estimating the load on a structure. Additional uncertainty exists in determining the magnitude and distribution of the mass of a structure and the load-deformation characteristics of the material from which the structure is made. No matter how "exact" is the analytic procedure for determining response, these inaccuracies in the parameters lead to an unreliable prediction of the behavior of the structure.

In order to interpret the significance of calculated responses, it is necessary to ascertain to what extent variations in these factors affect the response of the system. A broad understanding of the response of a structure to various patterns of impulsive load is required before one can investigate the effect on the response of deviations in the forcing function parameters and in the properties of the structure. The objectives of this investigation are to study the response of a simple structure to dynamic disturbances and to

determine the effect on that response of variations in the parameters of load and in the characteristics of the structure.

For this fundamental study the force functions and the structure considered are ones which can be defined by simple mathematical expressions. The impulsive load patterns include those blast and impact disturbances described by a step-pulse function or a triangular-pulse function. The structure under study is one which can be represented by an undamped, single-degree-of-freedom system with elasto-plastic resistance to motion.

The descriptions of the load and the structure involve a number of parameters. For a specific impulse pattern the average applied force and the duration of the load are the force parameters considered. The mass of the structure, the rigidity, and the elasto-plastic properties of the material from which the structure is made comprise the characteristics of the structure. Complete investigation of the problem requires consideration of the entire range of these parameters. To this end, the results of the analysis used are presented in dimensionless form.

The problem described lends itself to classical methods of analysis. To avoid errors introduced by numerical procedures, an analytic procedure is employed for determining the response of the system, its maximum response, and the influence of the problem parameters on that maximum response.

B. NOTATION

The terms used are defined when they first appear in the text. They are assembled here for convenience in reference.

K = spring stiffness

M = mass

X = displacement of mass

X_m = maximum displacement of mass

X_s = "static" displacement of mass = P/K

X_1 = P_1/K

X_y = yield displacement of spring = Q_y/K

X' = velocity of mass = dX/dt

X'' = acceleration of mass = d^2X/dt^2

$Q(X)$ = spring force-displacement function

Q_y = yield strength of spring

T = period of system = $2\pi\sqrt{M/K}$

ω = circular frequency of system = $\sqrt{K/M}$ = $2\pi/T$

$P(t)$ = force-time function

P_1 = average applied load

$P\alpha$ = maximum applied load

t = time

t_1 = duration of applied load

αt_1 = time of peak load

t_m = time of maximum displacement

t_y = time of first yielding

β = ratio of yield strength to average applied load = Q_y/P_1

$\Delta X_m/X_m$ = relative change in maximum displacement

$\Delta K/K$ = relative change in spring stiffness

$\Delta M/M$ = relative change in mass

$\Delta t_1/t_1$ = relative change in duration of load

$\Delta P_1/P_1$ = relative change in average applied load

$\Delta Q_y/Q_y$ = relative change in yield strength

C_K, C_M, C_t, C_P, C_Q = influence factors

II. RESPONSE OF THE STRUCTURE

A. DEFINITION OF PROBLEM

1. Governing equation of motion.

Physically, the undamped, single-degree-of-freedom system represents a mass supported by a spring which has no mass as shown in Fig. 1. When subjected to a dynamic disturbance, the structure behaves according the equation of motion:

$$M\ddot{X} + Q(X) = P(t), \tag{1}$$

where M is the mass; $\ddot{X}$, the acceleration of the mass; X, the displacement of the mass; Q(X), the resisting force in the spring; and P(t), the impulsive load function. In this study the mass is considered to be at rest with no initial displacement when the load is applied:

$$\text{at} \quad t = 0, \; X = \dot{X} = 0, \tag{2}$$

where t denotes time and $\dot{X}$ is the velocity of the mass.

2. Resistance of structure to motion.

The spring of the structure has "ideal" elasto-plastic characteristics: an initial linear resistance to motion before yielding, followed by a constant resistance until maximum displacement is reached. (Fig. 2). The elastic resisting force of the spring is directly proportional to the displacement:

$$Q(X) = KX \qquad \text{for} \qquad X \leqq X_y, \tag{3-a}$$

where K is a constant spring stiffness and X_y is the yield displacement of the spring. When the response is greater than the yield deflection, the spring resistance is constant:

$$Q(X) = K\, X_y = Q_y \quad , \quad \text{for } X \geqq X_y. \tag{3-b}$$

3. Forcing functions.

The response of the system to several patterns of impulsive loads is investigated. In order to compare the response of a particular structure to different types of exciting forces, the total impulse is held constant:

$$\int_o^{t_1} P(t)\, dt = P_1 t_1 \ , \tag{4}$$

P_1 being the average applied load and t_1, the duration of the applied load. First, study is made of the response due to a step-pulse function:

$$P(t) = P_1 \quad , \quad \text{for } o \leq t \leq t_1 \ ; \tag{5-a}$$

$$P(t) = 0 \quad , \quad \text{for } t_1 \leq t \ . \tag{5-b}$$

(Fig. 3-a). Next, consideration is given to the response of the system to a triangular forcing function with an initial peak force:

$$P(t) = 2P_1\, (1 - t/t_1) \quad , \quad \text{for } o \leq t \leq t_1 \ ; \tag{6-a}$$

$$P(t) = 0 \quad , \quad \text{for } t_1 \leq t. \tag{6-b}$$

(Fig. 3-b). In addition to these two impulse patterns, the triangular forcing functions with a terminal and an intermediate maximum force are considered. (Figs. 3-c and 3-d). The terminal-peak exciting force is defined:

$$P(t) = 2P_1\, (t/t_1) \quad , \quad \text{for } o \leq t \leq t_1 \tag{7-a}$$

$$P(t) = 0 \quad , \quad \text{for } t_1 \leq t. \tag{7-b}$$

The intermediate-peak impulsive load represents the general triangular forcing function:

$$P(t) = 2P_1 \, (t/\alpha t_1) \qquad , \text{ for } o \leq t \leq \alpha t_1 \; ; \tag{8-a}$$

$$P(t) = 2P_1 \, (t_1 - t) \, / \, (1 - \alpha) \, t_1 \; , \text{ for } \alpha t_1 \leq t \leq t_1 ; \tag{8-b}$$

$$P(t) = 0 \qquad , \text{ for } t_1 \leq t ; \tag{8-c}$$

where αt_1 defines the time at which the maximum load occurs ($0 \leq \alpha \leq 1$). The patterns of force described above are considered to represent simplified blast and impact disturbances normally encountered.

B. SOLUTION FOR RESPONSE

1. General response.

With the analytic procedure the general solution to the governing differential equation of motion (equation 1) is found:

$$X/X_y = A\cos\omega t + B\sin\omega t + P(t)/\beta P_1 , \quad \text{for } X \leq X_y \text{ and } P(t) \text{ linear;} \tag{9-a}$$

$$X/X_y = \frac{\omega^2}{\beta} \int_o^t \int_o^t P(t)/P_1 \, dt \quad -1/2\omega^2 t^2 + C\omega t + D, \quad \text{for } X_y \leq X \; ; \tag{9-b}$$

where A, B, C, and D denote constants of integration determined from the initial conditions of velocity and displacement, ω represents the circular frequency of the system, and β is the ratio of the yield strength to the average applied load:

$$\omega = \sqrt{K/M} \; ; \tag{10}$$

$$\beta = Q_y/P_1 \; . \tag{11}$$

The solution appears in dimensionless form. The quantities ω, t_1, and β

together with the pattern of the forcing function are sufficient to determine the behavior of a range of structures with respect to the yield deflection.

2. Maximum response.

By setting the velocity of the mass equal to zero, the time, t_m, at which the response is a maximum is found. Substitution of t_m in the response equation yields an expression for maximum deflection in terms of the yield displacement. These analytic formulae of maximum deflection for each type of load considered appear in the appendix.

The graphs (Figs. 4-7 inclusive) which represent the maximum response furnish a better picture of the system's behavior. For various ratios of yield strength to average applied load, β, the maximum response with respect to the yield deflection, X_m/X_y, is given in terms of the duration of load and the period of the structure, t_1/T. These charts show the relations between the duration of the load, t_1, the time to attain yielding, t_y, and the time of maximum deflection, t_m. An accurate estimate of the maximum response of a simple structure to specific types of load is directly obtainable from these response graphs provided the forcing function and the characteristics of the structure are known.

The type of impulsive excitation may, or may not, affect the maximum response of the structure. In the so-called impulse region when the duration of load is short in comparison with the period of the structure, it is of interest to note that the pattern of load does not significantly affect the maximum response, other things being equal. (Fig. 8). In contrast to this, the impulse pattern may greatly influence the maximum deflection when the load is applied very slowly relative to the system's period. (Fig. 9).

Of the forcing functions studied, the triangular pulse with an initial peak force produces the most critical response in the long-duration region; the response to a step-pulse function is the smallest in magnitude.

The maximum displacement curves pictured in the graphs also illustrate to what extent a change in one parameter, the duration of load t_1 for instance, affects the maximum response. In Fig. 5-a, which represents the response of the structure to an initial peak force, for β equal to 1.0 a twenty percent change in t_1 from 0.5T to 0.6T produces a thirty percent change in maximum response from 3.85 X_y to 5.00 X_y. It is not unreasonable to expect an error in certain parameters to result in an error three or more times as large in the maximum response.

3. Approximate maximum response.

In addition to the "exact" expressions for response it is desirable to obtain approximate solutions for the behavior of the system. Generally, there are two cases of particular interest: one in which the duration of the applied load is less than the time at which the maximum response occurs; the other in which the load terminates long after the maximum response is attained.

The spring in the first case responds inelastically early in the history of the system's behavior before the velocity of the mass is appreciable. Hence, it is reasonable to assume that the elastic response of the structure is negligible; that is, the spring has an initial displacement equal to the yield deflection, and the mass is at rest when the load is applied:

at $t = 0$ $\qquad Q(X) = KX_y = Q_y \;;$ (3-b)

$X = X_y \;;$ (2-a)

$\dot{X} = 0 \,.$ (2)

The general solution to this problem is of the form of equation (9-b), where, as before, the constants C and D are determined from the initial conditions for velocity and displacement. The maximum response is found in the same manner as previously described for the more rigorous solution. These expressions for maximum response are included in the appendix. How closely this approximate response agrees with the actual response in the impulse region is illustrated in Fig. 10, for the case of the step-pulse function. Similarly, for the various triangular forcing functions the agreement between approximate and exact maximum responses is quite good.

When the maximum deflection occurs long before the load ceases, the time at which the response is a maximum may not significantly exceed the time at which the spring first yields. For this reason, neglecting the elastic response of the system is not an accurate assumption for this case. A better approach to determining the approximate response is to take the limit of the "exact" solutions for maximum displacement allowing the duration of the load, t_1, to go to infinity. The expressions which result define the maximum response for a duration of load which is long in comparison with the period of the structure. The approximate solutions thus obtained are exact for the step-pulse function but, at best, are only indicative of the magnitude of the maximum response for the triangular pulse functions as shown in Fig. 11.

III. SENSITIVITY OF THE RESPONSE OF THE STRUCTURE

A. DETERMINATION OF SENSITIVITY

The analytic expressions for maximum response are used to determine the influence of the problem parameters on that response. This influence

may be defined as the change in maximum response due to a change in the parameter considered. If the deviation in that parameter is reduced to an infinitesimal, the partial derivative of the maximum displacement with respect to that parameter represents the influence factor sought. Then the summed effect of changes in every parameter on the maximum response is the total differential of the maximum displacement:

$$\Delta X_m = \frac{\partial X_m}{\partial P_1} \Delta P_1 + \frac{\partial X_m}{\partial Q_y} \Delta Q_y + \frac{\partial X_m}{\partial t_1} \Delta t_1 + \frac{\partial X_m}{\partial K} \Delta K + \frac{\partial X_m}{\partial M} \Delta M \, . \tag{12}$$

Dividing this expression by the maximum displacement, X_m, gives the dimensionless relation:

$$\frac{\Delta X_m}{X_m} = C_P \frac{\Delta P_1}{P_1} + C_Q \frac{\Delta Q_y}{Q_y} + C_t \frac{\Delta t_1}{t_1} + C_K \frac{\Delta K}{K} + C_M \frac{\Delta M}{M} \tag{12-a}$$

where C_P, C_Q, C_t, C_K, and C_M are a measure of the sensitivity of the response of the structure:

$$C_P = \frac{\partial X_m}{\partial P_1} \cdot \frac{P_1}{X_m} \, , \tag{13}$$

and similarly for the other parameters. These influence factors are charted in Figs. 12-15 for the step-pulse function and the initial-peak triangular forcing function. For each type of impulse pattern studied the analytic expressions for the influence factors are included in the appendix. Because of the dimensionless form of the formulae for maximum response, certain relations exist between the influence factors:

$$C_P = 1 - C_Q \, ; \tag{14-a}$$

$$C_K = 1/2\, C_t - 1 \, ; \tag{14-b}$$

$$C_M = -\, 1/2\, C_t \, . \tag{14-c}$$

This allows the influence factors C_P, C_K, and C_M to be defined in terms of C_Q and C_t.

B. USE OF INFLUENCE FACTORS

The influence factors represented in the graphs and in the analytic expressions given in the appendix may be used to find the change in maximum response, $\Delta X_m/X_m$, due to a change in each parameter. For a particular structure and a specified forcing function the magnitudes of the maximum response and the corresponding influence factors are obtained either from the graphs or from the analytic expressions. When substituted into equation (12-a), these quantities and the magnitudes of the relative changes whose effect is to be examined yield the change in the maximum response.

Although this procedure is exact for only infinitesimal changes in the parameters, equation (12-a) may be used with a certain degree of accuracy for finite changes in the parameters. It can be shown that the error involved by considering finite variations is of a higher order than those variations. A discussion on the limitations of the magnitude of the change taken in the parameters is reserved to the appendix. Generally speaking, the error involved is not significant provided the relative variations in the parameters are kept in the neighborhood of ten percent absolute. Larger changes than these can be made step-wise without introducing appreciable error; that is, by considering the variations in ten percent steps, or less, and replacing the old structure by the new at each step, the change in maximum response due to large changes in the parameters is found.

C. SIGNIFICANCE OF RESULTS

The graphs for the influence factors illustrate the order of importance of the changes in the various parameters. In the impulse region the average applied load is generally the most critical in its effect on the

maximum response with the duration of load second in importance. The average applied load again has the greatest influence on the maximum response in the long-duration region; there the yield strength is second in importance.

That an error in a parameter may result in an error in the maximum response several times as large is demonstrated by the graphs representing the influence factors. In the long-duration region the influence factor for average load, for instance, may be as high as 8 or more. Then too, a variation in a parameter may have little or no effect on the maximum response. The duration of load, for example, has little effect in the long-duration regions for β greater than 3.0. The range over which the influence factors may vary accounts for some of the discrepancies observed in practice.

IV. SUMMARY OF RESULTS

To justify using rigorous methods of analysis for elasto-plastic structures subjected to dynamic loads, it is necessary to determine accurately the magnitudes of the characteristics of the structure and the forcing function parameters. If these quantities are not precisely known, the effect of an error in a parameter on the maximum response may be determined by the method previously described.

Presented in graphical form, the results of the solution for the maximum response of a simple structure to dynamic loads may be used in practice for predicting the behavior of structures. The approximate expressions for maximum response given in the appendix are accurate for estimating purpose in the impulse region only. In certain cases the response in the first mode of a multi-degree-of-freedom system can be **estimated**

from the results here presented. It must be kept in mind, however, that while these exact and approximate results are accurate for a single-degree-of-freedom system, the extension of their use to application to other structural systems is, at best, approximate.

In design and analysis the magnitude of some of the parameters is often a matter of arbitrary selection. When this condition exists, the use of a rigorous method of analysis in predicting the behavior of the structure is meaningless. The results of this investigation give support to the belief that in such a case it is more appropriate to employ numerical or other approximate procedures which are in keeping with the quality of the data available.

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APPENDIX

A. ANALYTIC EXPRESSIONS FOR MAXIMUM RESPONSE AND INFLUENCE FACTORS

1. Step pulse.

(A). $0 \leq t_m \leq t_1 \leq t_y$.

(1). $\omega t_m = (2n-1)\pi \quad , \qquad n = 1, 2, 3, \ldots\ldots$

(2). $X_m / X_y = 2/\beta$.

(3). $C_Q = 0$.

(4). $C_t = 0$.

(B). $0 \leq t_1 \leq t_m \leq t_y$.

(1). $\omega t_m = \tan^{-1} \left(\frac{\sin \omega t_1}{\cos \omega t_1 - 1} \right)$.

(2). $X_m / X_y = \frac{2}{\beta} \sin \tfrac{1}{2} \omega t_1$.

(3). $C_Q = 0$.

(4). $C_t = \frac{\omega t_1}{2} \cot \tfrac{1}{2} \omega t_1$.

(C). $0 \leq t_1 \leq t_y \leq t_m$.

(1). $\cos(\omega t_y - \omega t_1) = \cos \omega t_y + \beta$.

(2). $\omega t_m = \omega t_y + \frac{1}{\beta} \left\{ \sin \omega t_y - \sin(\omega t_y - \omega t_1) \right\}$.

(3). $X_m / X_y = \tfrac{1}{2} + \frac{1}{\beta^2} (1 - \cos \omega t_1)$.

(4). $C_Q = -1 + \frac{1}{X_m / X_y}$.

(5). $C_t = \frac{\omega t_1 \sin \omega t_1}{\beta^2} \left(\frac{1}{X_m / X_y} \right)$.

(D). $0 \leq t_y \leq t_m \leq t_1$.

(1). $\cos \omega t_y = 1 - \beta$.

(2). $\omega t_m = \omega t_y - \dfrac{\sin \omega t_y}{1-\beta}$.

(3). $X_m / X_y = \dfrac{1}{2(\beta-1)}$.

(4). $C_Q = \dfrac{\beta-2}{\beta-1}$.

(5). $C_t = 0$.

(E). $\underline{0 \leq t_y \leq t_1 \leq t_m}$.

(1). $\cos \omega t_y = 1-\beta$.

(2). $\omega t_m = \dfrac{1}{\beta}\left\{\omega t_1 + \sin \omega t_y - (1-\beta)\omega t_y\right\}$.

(3). $X_m / X_y = \dfrac{2+\beta}{2\beta} + \dfrac{1}{\beta}\left\{\dfrac{2-\beta}{\beta}\right\}^{1/2}(\omega t_1 - \omega t_y)$
$+ \dfrac{1-\beta}{2\beta^2}(\omega t_1 - \omega t_y)^2$.

(4). $C_Q = -1 + \dfrac{1}{X_m/X_y}\left\{1 - \dfrac{1}{2\beta}(\omega t_1 - \omega t_y)^2\right\}$.

(5). $C_t = \dfrac{\omega t_1}{\beta X_m/X_y}\left[\left\{\dfrac{2-\beta}{\beta}\right\}^{1/2} + \dfrac{1-\beta}{\beta}(\omega t_1 - \omega t_y)\right]$.

2. Initial-peak triangular pulse.

(A). $\underline{0 \leq t_m \leq t_1 \leq t_y}$.

(1). $\omega t_m = \sin^{-1}\left\{\dfrac{2\omega t_1}{1+\omega^2 t_1^2}\right\}$.

(2). $X_m / X_y = \dfrac{2}{\beta}\left\{2 - \dfrac{\omega t_m}{\omega t_1}\right\}$.

(3). $C_Q = 0$.

(4). $C_t = -1 + \dfrac{4}{\beta X_m/X_y}\left\{\omega^2 t_1^2 + \dfrac{1}{1+\omega^2 t_1^2}\right\}$.

(B). $\underline{0 \leq t_1 \leq t_m \leq t_y}$.

(1). $\omega t_m = \tan^{-1}\left\{\dfrac{1-\cos \omega t_1}{\sin \omega t_1 - \omega t_1}\right\}$.

(2). $X_m/X_y = \dfrac{2}{\beta\omega t_1}\left\{2(1-\cos\omega t_1-\omega t_1\sin\omega t_1)+\omega^2 t_1^2\right\}^{1/2}$.

(3). $C_Q = 0$.

(4). $C_t = -1+\left\{\dfrac{2}{\beta X_m/X_y}\right\}^2\left\{1-\cos\omega t_1\right\}$.

(C). $\underline{0 \leq t_1 \leq t_y \leq t_m}$.

(1). $\dfrac{\beta}{2}\omega t_1 = \sin\omega t_y - \omega t_1\cos\omega t_y - \sin(\omega t_y-\omega t_1)$.

(2). $\omega t_m = \omega t_y + \dfrac{2}{\beta\omega t_1}\left\{\cos\omega t_y+\omega t_1\sin\omega t_y-\cos(\omega t_y-\omega t_1)\right\}$.

(3). $X_m/X_y = 1+\dfrac{1}{2}(\omega t_m-\omega t_y)^2$.

(4). $C_Q = \dfrac{-1}{2X_m/X_y}(\omega t_m-\omega t_y)^2$.

(5). $C_t = -1-\dfrac{1}{2X_m/X_y}(\omega t_m-\omega t_y)^2+\dfrac{4}{\beta^2 X_m/X_y}(1-\cos\omega t_1)$.

(D). $\underline{0 \leq t_y \leq t_m \leq t_1}$.

(1). $\omega t_1 = \dfrac{\omega t_y-\sin\omega t_y}{1-\beta/2-\cos\omega t_y}$.

(2). $\omega t_m = (1-\beta/2)\omega t_1+\left[\begin{aligned}&\{(1-\beta/2)\omega t_1-\omega t_y\}^2\\&+2\{\omega t_1\sin\omega t_y+\cos\omega t_y-1\}\end{aligned}\right]^{1/2}$.

(3). $X_m/X_y = 1+\dfrac{2}{3\beta\omega t_1}\{\omega t_m-\omega t_y\}^3+\dfrac{1}{2}\{\omega t_m-\omega t_y\}^2$
$$-\frac{1}{\beta\omega t_1}\{\omega t_1-\omega t_y\}\{\omega t_m-\omega t_y\}^2 .$$

(4). $C_Q = -\dfrac{1}{2X_m/X_y}\{\omega t_m-\omega t_y\}^2$.

(5). $C_t = -1+\dfrac{1}{X_m/X_y}\left\{\begin{aligned}&-1/2(\omega t_m-\omega t_y)^2\\&+\frac{2}{\beta}\omega t_1\left[\sin\omega t_y+(\omega t_m-\omega t_y)\cos\omega t_y\right]\end{aligned}\right\}$.

(E). $\underline{0 \leq t_y \leq t_1 \leq t_m}$.

(1). $\omega t_1 = \dfrac{\omega t_y-\sin\omega t_y}{1-\beta/2-\cos\omega t_y}$.

(2). $\omega t_m = \omega t_4 + \dfrac{2}{\beta\omega t_1}\left\{\tfrac{1}{2}(\omega t_1 - \omega t_4)^2 + \omega t_1 \sin\omega t_4 + \cos\omega t_4 - 1\right\}.$

(3). $X_m/X_q = 1 + \dfrac{1}{2}(\omega t_m - \omega t_4)^2 - \dfrac{1}{3\beta\omega t_1}(\omega t_1 - \omega t_4)^3 .$

(4). $C_Q = \dfrac{-1}{2X_m/X_q}(\omega t_m - \omega t_4)^2.$

(5). $C_t = -1 + \dfrac{1}{X_m/X_q}\left[\begin{array}{l} -\tfrac{1}{2}(\omega t_m - \omega t_4)^2 \\ + \dfrac{4}{\beta^2}\left\{\begin{array}{l}\tfrac{1}{2}(\omega t_1 - \omega t_4)^2\cos\omega t_4 + 1 - \cos\omega t_4 \\ +(\omega t_1 - \omega t_4)\sin\omega t_4\end{array}\right\}\end{array}\right].$

3. Terminal-peak triangular pulse.

(A). $0 \le t_m \le t_1 \le t_4$.

(1). $\omega t_m = 2n\pi \quad , \qquad n = 1, 2, \ldots .$

(2). $X_m/X_q = \dfrac{2\omega t_m}{\beta\omega t_1} .$

(3). $C_Q = 0.$

(4). $C_t = -1.$

(B). $0 \le t_1 \le t_m \le t_4$.

(1). $\omega t_m = \tan^{-1}\left\{\dfrac{\omega t_1 \sin\omega t_1 - 1 + \cos\omega t_1}{\omega t_1 \cos\omega t_1 - \sin\omega t_1}\right\}.$

(2). $X_m/X_q = \dfrac{2}{\beta\omega t_1}\left\{\omega^2 t_1^2 + 2(1 - \omega t_1 \sin\omega t_1 - \cos\omega t_1)\right\}^{1/2}.$

(3). $C_Q = 0.$

(4). $C_t = -1 + \left\{\dfrac{2}{\beta X_m/X_q}\right\}^2\left\{1 - \cos\omega t_1\right\}.$

(C). $0 \le t_1 \le t_4 \le t_m$.

(1). $\dfrac{\beta}{2}\omega t_1 = \omega t_1 \cos(\omega t_4 - \omega t_1) + \sin(\omega t_4 - \omega t_1) - \sin\omega t_4.$

(2). $\omega t_m = \omega t_4 + \dfrac{2}{\beta\omega t_1}\left\{\cos(\omega t_4 - \omega t_1) - \omega t_1 \sin(\omega t_4 - \omega t_1) - \cos\omega t_4\right\}.$

(3). $X_m/X_y = 1 + 1/2\,(\omega t_m - \omega t_y)^2.$

(4). $C_Q = \dfrac{-1}{2X_m/X_y}(\omega t_m - \omega t_y)^2.$

(5). $C_t = -1 + \dfrac{1}{X_m/X_y}\left\{-1/2\,(\omega t_m - \omega t_y)^2 + \dfrac{4}{\beta^2}(1-\cos\omega t_1)\right\}.$

(D). $\underline{0 \le t_y \le t_m = t_1.}$

(1). $\dfrac{\beta}{2}\omega t_1 = \omega t_y - \sin\omega t_y.$

(2). $\omega t_m = \omega t_1.$

(3). $X_m/X_y = 1 + \dfrac{2}{\beta\omega t_1}\left\{\begin{array}{l}1/6\,(\omega t_1 - \omega t_y)^3 + 1/2\,(\omega t_1 - \omega t_y)^2 \sin\omega t_y \\ \quad + (\omega t_1 - \omega t_y)(1-\cos\omega t_y)\end{array}\right\}.$

(4). $C_Q = \dfrac{-1}{2X_m/X_y}(\omega t_1 - \omega t_y)^2.$

(5) $C_t = -1 + \dfrac{1}{X_m/X_y}\left[\begin{array}{l}-1/2\,(\omega t_1 - \omega t_y)^2 \\ \quad + \dfrac{2}{\beta}\left\{\begin{array}{l}1/2\,(\omega t_1 - \omega t_y)^2 + 1 - \cos\omega t_y \\ \quad + (\omega t_1 - \omega t_y)\sin\omega t_y\end{array}\right\}\end{array}\right].$

(E). $\underline{0 \le t_y \le t_1 \le t_m.}$

(1). $\dfrac{\beta}{2}\omega t_1 = \omega t_y - \sin\omega t_y.$

(2). $\omega t_m = \omega t_1 + \dfrac{2}{\beta\omega t_1}\left\{1/2\,(\omega t_1 - \omega t_y)^2 + (\omega t_1 - \omega t_y)\sin\omega t_y + 1 - \cos\omega t_y\right\}.$

(3). $X_m/X_y = 1 + 1/2\,(\omega t_m - \omega t_1)^2$

$\quad + \dfrac{2}{\beta\omega t_1}\left\{\begin{array}{l}1/6\,(\omega t_1 - \omega t_y)^3 + 1/2\,(\omega t_1 - \omega t_y)^2 \sin\omega t_y \\ \quad + (\omega t_1 - \omega t_y)(1-\cos\omega t_y)\end{array}\right\}.$

(4). $C_Q = \dfrac{-1}{2X_m/X_y}(\omega t_m - \omega t_y)^2.$

(5). $C_t = -1 + \dfrac{1}{X_m/X_y}\left\{-1/2\,(\omega t_m - \omega t_y)^2 + \dfrac{2}{\beta}\omega t_1(\omega t_m - \omega t_1)\right\}.$

4. Intermediate-peak triangular pulse.

(A). $\underline{0 \le t_m \le \alpha t_1 \le t_1 \le t_y.}$

(1). $\omega t_m = 2n\pi \quad ; \qquad n = 1, 2, 3 \cdots\cdots$

(2). $X_m/X_y = \dfrac{4n\pi}{\beta\alpha\omega t_1}$.

(3). $C_Q = 0$.

(4). $C_t = -1$.

(B). $\underline{0 \leq \alpha t_1 \leq t_m \leq t_1 \leq t_y}$.

(1). $$\cos \omega t_m = \frac{\alpha\{\cos\alpha\omega t_1 - (1-\alpha)\} + 2(1-\alpha)^{1/2}\sin\alpha\omega t_1 \left|\sin^{\alpha/2}\omega t_1\right|}{2 - 2\alpha + \alpha^2 - 2(1-\alpha)\cos\alpha\omega t_1}$$

(2). $$X_m/X_y = \frac{2}{\beta\alpha(1-\alpha)\omega t_1}\left\{2(1-\alpha)^{1/2}\left|\sin^{\alpha/2}\omega t_1\right| + \alpha(\omega t_1 - \omega t_m)\right\}.$$

(3). $C_Q = 0$.

(4). $$C_t = -1 + \frac{2}{\beta(1-\alpha)X_m/X_y}\left\{(1-\alpha)^{1/2}\cos^{\alpha/2}\omega t_1 + 1 - \frac{\partial\omega t_m}{\partial\omega t_1}\right\}.$$

$$\frac{\partial\omega t_m}{\partial\omega t_1} = \frac{1}{\cos\omega t_m\left(2-2\alpha+\alpha^2-2(1-\alpha)\cos\alpha\omega t_1\right)}\left\{\begin{array}{l}\alpha^2\cos\alpha\omega t_1 \\ -2\alpha(1-\alpha)\sin\alpha\omega t_1\sin\omega t_m \\ +2\alpha(1-\alpha)^{1/2}\sin^{\alpha/2}\omega t_1\sin^{\alpha/2}\omega t_1 \\ +\alpha(1-\alpha)^{1/2}\cos^{\alpha/2}\omega t_1(1-\alpha) \\ -\alpha(1-\alpha)^{1/2}\cos^{\alpha/2}\omega t_1\cos\alpha\omega t_1\end{array}\right\}$$

(C). $\underline{0 \leq \alpha t_1 \leq t_1 \leq t_m \leq t_y}$.

(1). $$\omega t_m = \tan^{-1}\left\{\frac{\cos\alpha\omega t_1 - \alpha\cos\omega t_1 - (1-\alpha)}{\alpha\sin\omega t_1 - \sin\alpha\omega t_1}\right\}.$$

(2). $$X_m/X_y = \frac{2}{\beta\alpha(1-\alpha)\omega t_1}\left\{\begin{array}{c}2(1-\alpha)(1-\cos\alpha\omega t_1 + \alpha\cos\omega t_1) \\ +2\alpha(\alpha - \cos(1-\alpha)\omega t_1)\end{array}\right\}^{1/2}.$$

(3). $C_Q = 0$.

(4). $$C_t = -1 + \frac{4}{\beta^2\alpha\omega t_1(1-\alpha)(X_m/X_y)^2}\left\{\sin\alpha\omega t_1 - \sin\omega t_1 + \sin(1-\alpha)\omega t_1\right\}.$$

(D). $\underline{0 \leq \alpha t_1 \leq t_1 \leq t_y \leq t_m}$.

(1). $$\frac{\beta\alpha}{2}(1-\alpha)\omega t_1 = \sin(\omega t_y - \alpha\omega t_1) - \alpha\sin(\omega t_y - \omega t_1) - (1-\alpha)\sin\omega t_y.$$

(2). $$\omega t_m = \omega t_y + \frac{2}{\beta\alpha(1-\alpha)\omega t_1}\cos(\omega t_y - \alpha\omega t_1) - \alpha\cos(\omega t_y - \omega t_1) - (1-\alpha)\cos\omega t_y.$$

(3). $$X_m/X_y = 1 + \tfrac{1}{2}(\omega t_m - \omega t_y)^2.$$

(4). $C_Q = -\dfrac{1}{2X_m/X_y}(\omega t_m - \omega t_y)^2.$

(5). $C_t = -2 + \dfrac{1}{X_m/X_y}\left[1 + \dfrac{4\{\sin\alpha\omega t_1 - \sin\omega t_1 + \sin(1-\alpha)\omega t_1\}}{\beta^2\alpha(1-\alpha)\omega t_1}\right]$

(E). $\underline{0 \leq \alpha t_1 \leq t_y \leq t_m \leq t_1}.$

(1). $\dfrac{\beta}{2}\alpha(1-\alpha)\omega t_1 = \alpha(\omega t_1 - \omega t_y) - (1-\alpha)\sin\omega t_y + \sin(\omega t_y - \alpha\omega t_1).$

(2). $$\omega t_m = \omega t_1\left(1 - \beta/2\,(1-\alpha)\right) + \left[\begin{array}{l}\{\omega t_1(1-\beta/2\,(1-\alpha)) - \omega t_y\}^2 \\ + \dfrac{2}{\alpha}\{\cos(\omega t_y - \alpha\omega t_1) - (1-\alpha)\cos\omega t_y - \alpha\}\end{array}\right]^{1/2}$$

(3). $X_m/X_y = 1 + \dfrac{2}{3\beta(1-\alpha)\omega t_1}(\omega t_m - \omega t_y)^3 - \dfrac{1}{\beta(1-\alpha)\omega t_1}(\omega t_m - \omega t_y)^2(\omega t_1\{1-\beta/2\,(1-\alpha)\} - \omega t_y)$

(4). $C_Q = -\dfrac{1}{2X_m/X_y}(\omega t_m - \omega t_y)^2.$

(5). $$C_t = -1 + \frac{1}{X_m/X_y}\left[\begin{array}{l}-1/2\,(\omega t_m - \omega t_y)^2 \\ + \dfrac{1}{\beta(1-\alpha)}\left\{\begin{array}{l}(\omega t_m - \omega t_y)^2 + 2(1-\cos(\omega t_y - \alpha\omega t_1)) \\ +2(\omega t_m - \omega t_y)\sin(\omega t_y - \alpha\omega t_1)\end{array}\right\}\end{array}\right]$$

(F). $\underline{0 \leq \alpha t_1 \leq t_y \leq t_1 \leq t_m}.$

(1). $\dfrac{\beta}{2}\alpha(1-\alpha)\omega t_1 = \alpha(\omega t_1 - \omega t_y) - (1-\alpha)\sin\omega t_y + \sin(\omega t_y - \alpha\omega t_1)$

(2). $$\omega t_m = \omega t_y + \frac{2}{\beta\alpha(1-\alpha)\omega t_1}\left\{\begin{array}{l}\alpha/2\,(\omega t_1 - \omega t_y)^2 \\ +\cos(\omega t_y - \alpha\omega t_1) - (1-\alpha)\cos\omega t_y - \alpha\end{array}\right\}.$$

(3). $X_m/X_y = 1 + 1/2\,(\omega t_m - \omega t_y)^2 - \dfrac{1}{3\beta(1-\alpha)\omega t_1}(\omega t_1 - \omega t_y)^3.$

(4). $C_Q = \dfrac{-1}{2X_m/X_y}(\omega t_m - \omega t_y)^2.$

(5). $$C_t = -1 + \frac{1}{X_m/X_y}\left[\begin{array}{l}-1/2\,(\omega t_m - \omega t_y)^2 \\ + 2/\beta\,(\omega t_m - \omega t_y)\left\{\begin{array}{l}\beta/2\,\alpha\omega t_1 + \sin\omega t_y \\ + \omega t_1 - \omega t_y\end{array}\right\} \\ + \dfrac{1}{\beta(1-\alpha)}\left\{\begin{array}{l}2(1-\cos(\omega t_y - \alpha\omega t_1)) \\ - (\omega t_1 - \omega t_y)^2\end{array}\right\}\end{array}\right]$$

(G). $\underline{0 \leq t_y \leq t_m = \alpha t_1 \leq t_1}.$

(1) $\dfrac{\beta}{2}\alpha\omega t_1 = \omega t_y - \sin\omega t_y.$

(2). $\omega t_m = \alpha\omega t_1$.

(3). $$X_m/X_y = 1 + \frac{2}{\beta\alpha\omega t_1}\left\{\begin{aligned}&1/6\,(\alpha\omega t_1 - \omega t_y)^3 + 1/2\,(\alpha\omega t_1 - \omega t_y)^2 \sin\omega t_y\\ &+ (\alpha\omega t_1 - \omega t_y)(1-\cos\omega t_y)\end{aligned}\right\}$$

(4). $$C_\varphi = \frac{-1}{2X_m/X_y}(\alpha\omega t_1 - \omega t_y)^2.$$

(5). $$C_t = -1 + \frac{1}{X_m/X_y}\left[\begin{aligned}&-1/2\,(\alpha\omega t_1 - \omega t_y)^2\\ &+\frac{2}{\beta}\left\{\begin{aligned}&1/2\,(\alpha\omega t_1 - \omega t_y)^2\\ &+(\alpha\omega t_1 - \omega t_y)\sin\omega t_y\\ &+1-\cos\omega t_y\end{aligned}\right\}\end{aligned}\right].$$

(H). $\underline{0 \le t_y \le \alpha t_1 \le t_m \le t_1.}$

(1). $\frac{\beta}{2}\alpha\omega t_1 = \omega t_y - \sin\omega t_y$.

(2). $$\omega t_m = \omega t_1(1 - \beta/2(1-\alpha)) + \left[\begin{aligned}&\{\omega t_1(1-\alpha)(1-\beta/2)\}^2\\ &+\frac{2}{\alpha}(1-\alpha)\left\{\begin{aligned}&1/2\,(\alpha\omega t_1 - \omega t_y)^2 - \cos\omega t_y\\ &+(\alpha\omega t_1 - \omega t_y)\sin\omega t_y + 1\end{aligned}\right\}\end{aligned}\right]^{1/2}.$$

(3). $$\begin{aligned}X_m/X_y = 1 &+ \frac{2}{3\beta(1-\alpha)\omega t_1}(\omega t_m - (1-\beta/2(1-\alpha))\omega t_1)^3\\ &- \frac{(1-\beta/2)}{3\beta}\{(1-\alpha)(1-\beta/2)\omega t_1\}^2\\ &+ \frac{(1-\beta/2)}{\beta}\{\omega t_m - (1-\beta/2(1-\alpha))\omega t_1\}^2\\ &+ \frac{2}{\beta\alpha\omega t_1}\left\{\begin{aligned}&1/6\,(\alpha\omega t_1 - \omega t_y)^3 + (\alpha\omega t_1 - \omega t_y)(1-\cos\omega t_y)\\ &+1/2\,(\alpha\omega t_1 - \omega t_y)^2\sin\omega t_y\end{aligned}\right\}\end{aligned}$$

(4). $$C_\varphi = -\frac{1}{2X_m/X_y}(\omega t_m - \omega t_y)^2.$$

(5). $$C_t = -1 + \frac{1}{X_m/X_y}\left\{\begin{aligned}&-1/2\,(\omega t_m - \omega t_y)^2\\ &+\frac{1}{\beta(1-\alpha)}(\omega t_m - \alpha\omega t_1)^2\end{aligned}\right\}.$$

(I). $\underline{0 \le t_y \le \alpha t_1 \le t_1 \le t_m.}$

(1) $\frac{\beta}{2}\alpha\omega t_1 = \omega t_y - \sin\omega t_y$.

(2). $\omega t_m = \alpha\omega t_1 + \dfrac{2}{\beta\alpha\omega t_1}\left\{ \begin{array}{l} \tfrac{1}{2}\alpha(1-\alpha)\omega^2 t_1^2 + \tfrac{1}{2}(\alpha\omega t_1 - \omega t_y)^2 \\ + (\alpha\omega t_1 - \omega t_y)\sin\omega t_y + 1 - \cos\omega t_y \end{array} \right\}.$

(3). $X_m/X_y = 1 + \tfrac{1}{2}(\omega t_m - \alpha\omega t_1)^2 - \dfrac{(1-\alpha)\omega t_1)^2}{3\beta}$

$+ \dfrac{2}{\beta\alpha\omega t_1}\left\{ \begin{array}{l} \tfrac{1}{6}(\alpha\omega t_1 - \omega t_y)^3 + \tfrac{1}{2}(\alpha\omega t_1 - \omega t_y)^2 \sin\omega t_y \\ + (\alpha\omega t_1 - \omega t_y)(1 - \cos\omega t_y) \end{array} \right\}.$

(4). $C_\varphi = \dfrac{-1}{2X_m/X_y}(\omega t_m - \omega t_y)^2.$

(5). $C_t = -1 + \dfrac{1}{X_m/X_y}\left[\begin{array}{l} -\tfrac{1}{2}(\omega t_m - \omega t_y)^2 \\ + \dfrac{1}{\beta}\left\{ 2\omega t_1(\omega t_m - \alpha\omega t_1) - (1-\alpha)\omega^2 t_1^2 \right\} \end{array} \right].$

B. APPROXIMATE EXPRESSIONS FOR MAXIMUM RESPONSE.

1. Step pulse.

(A). Impulse region.

(1). $\omega t_m \cong \dfrac{\omega t_1}{\beta}.$

(2). $X_m/X_y \cong 1 + \left\{\dfrac{\omega t_1}{\beta}\right\}^2 \left\{\tfrac{1}{2} - \beta/2\right\}.$

(B). Long-duration region.

(1). $\cos\omega t_y = 1 - \beta.$

(2). $\omega t_m = \omega t_y - \tan\omega t_y.$

(3). $X_m/X_y = \dfrac{1}{2(\beta - 1)}.$

2. Initial-peak triangular pulse.

(A). Impulse region.

(1). $\omega t_m \cong \dfrac{\omega t_1}{\beta}.$

(2). $X_m/X_y \cong 1 + \left(\dfrac{\omega t_1}{\beta}\right)^2 \left(\tfrac{1}{2} - \tfrac{1}{3}\beta\right).$

(B). Long-duration region.

(1). $\omega t_m \cong \omega t_1(2-\beta)$.

(2). $X_m/X_y \cong 1 + \frac{1}{6\beta}(2-\beta)^3\omega^2 t_1^2$.

3. Terminal-peak triangular pulse.

(A). Impulse region.

(1). $\omega t_m \cong \frac{\omega t_1}{\beta}$.

(2). $X_m/X_y \cong 1 + \left(\frac{\omega t_1}{\beta}\right)^2\left(1/2 - 2\beta/3\right)$.

(B). Long-duration region.

(1). $\omega t_m \cong \omega t_1\left\{1 + \frac{(2-\beta)^2}{4\beta}\right\}$

(2). $X_m/X_y \cong 1 + \left\{\frac{(2-\beta)\omega t_1}{4\beta}\right\}^2\{2-\beta\}\{1+\beta/6\}$.

4. Intermediate-peak triangular pulse.

(A). Impulse region.

(1). $\omega t_m \cong \frac{\omega t_1}{\beta}$.

(2). $X_m/X_y \cong 1 + \left\{\frac{\omega t_1}{\beta}\right\}^2\left\{1/2 - \frac{\beta(1+\alpha)}{3}\right\}$.

(B). Long-duration region.

(1). $\omega t_m \cong \omega t_1\left\{1 - \beta/2(1-\alpha) + (1-\beta/2)(1-\alpha)^{1/2}\right\}$.

(2). $X_m/X_y \cong 1 + \frac{1}{12\beta}(2-\beta)^3\omega^2 t_1^2(2-\alpha)$.

C. ERROR INTRODUCED BY FINITE CHANGES IN PARAMETERS

If the error in an influence factor due to a finite change in a parameter is defined as:

$$\text{error} = \frac{C_p' - C_p}{C_p} \,, \qquad (15)$$

where $$C_p = \frac{\Delta X_m}{X_m} \cdot \left(\frac{p}{\Delta p}\right) ,$$

and $$C_p' = \lim_{\Delta p \to 0} \frac{\Delta X_m}{\Delta p}\left(\frac{p}{X_m}\right) = \frac{\partial X_m}{\partial p} \left(\frac{p}{X_m}\right) ,$$

the bounds may be set on that error such that:

$$a \leq e \leq b \quad . \qquad \text{for} \quad a \leq b \; .$$

or $$a \leq \frac{C_p'}{C_p} - 1 \leq b \quad ;$$

$$a + 1 \leq \frac{C_p'}{C_p} \leq b + 1 \quad . \qquad (16)$$

This restriction limits the magnitude of the change in the parameter under consideration to:

$$a + 1 \leq \frac{C_p'}{\Delta X_m/X_m} \left(\frac{\Delta p}{p}\right) \leq b + 1 \quad , \qquad (16\text{-a})$$

where a and b define the desired accuracy, C_p' is the value for the influence factor in the region under study, and $\Delta X_m/X_m$ is the allowable relative change in maximum deflection. For example, if $b = -a = 0.1$, $C_p' = 1.8$, and $\Delta X_m/X_m = 0.2$, then:

$$0.1 \leq \frac{\Delta p}{p} \leq 0.1222222 \quad .$$

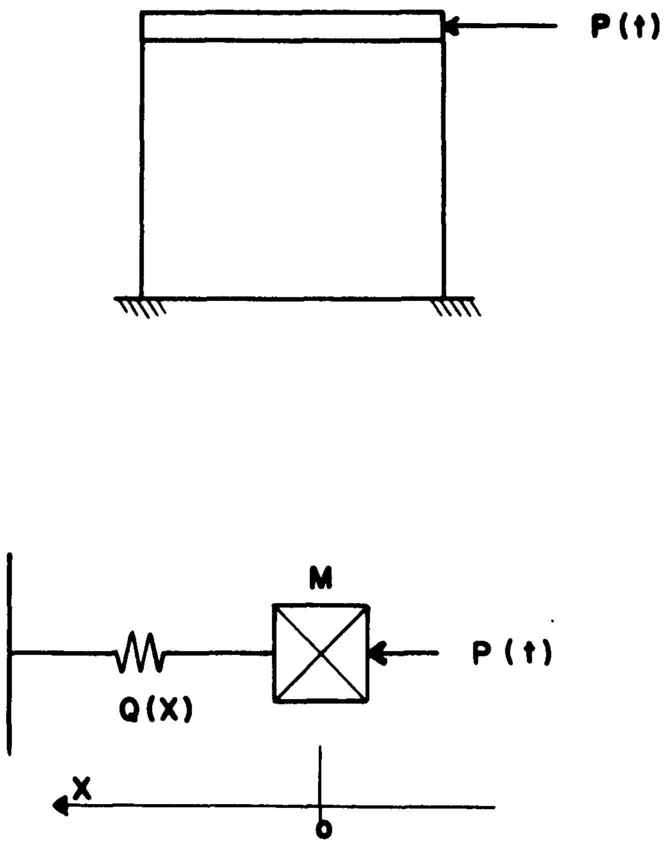

FIG. 1 SINGLE-DEGREE-OF-FREEDOM SYSTEM

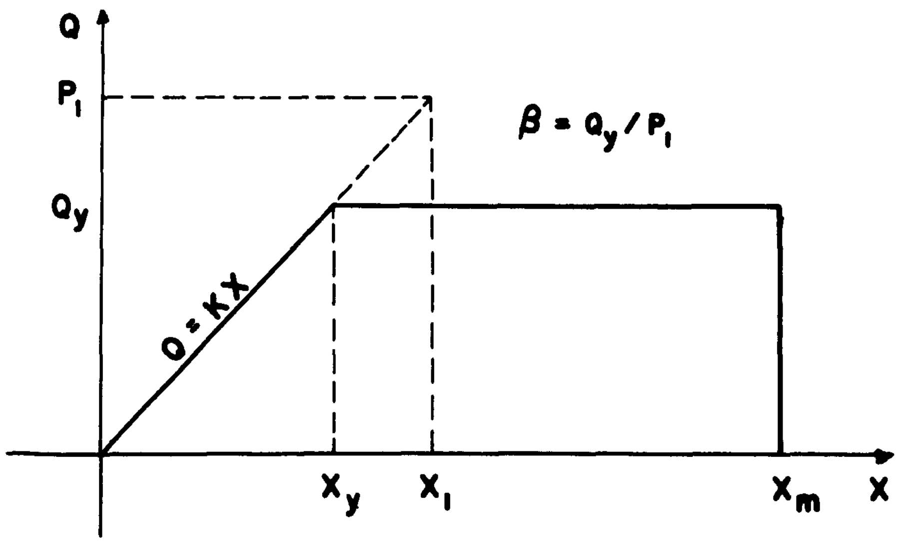

FIG. 2 SPRING FORCE-DISPLACEMENT FUNCTION

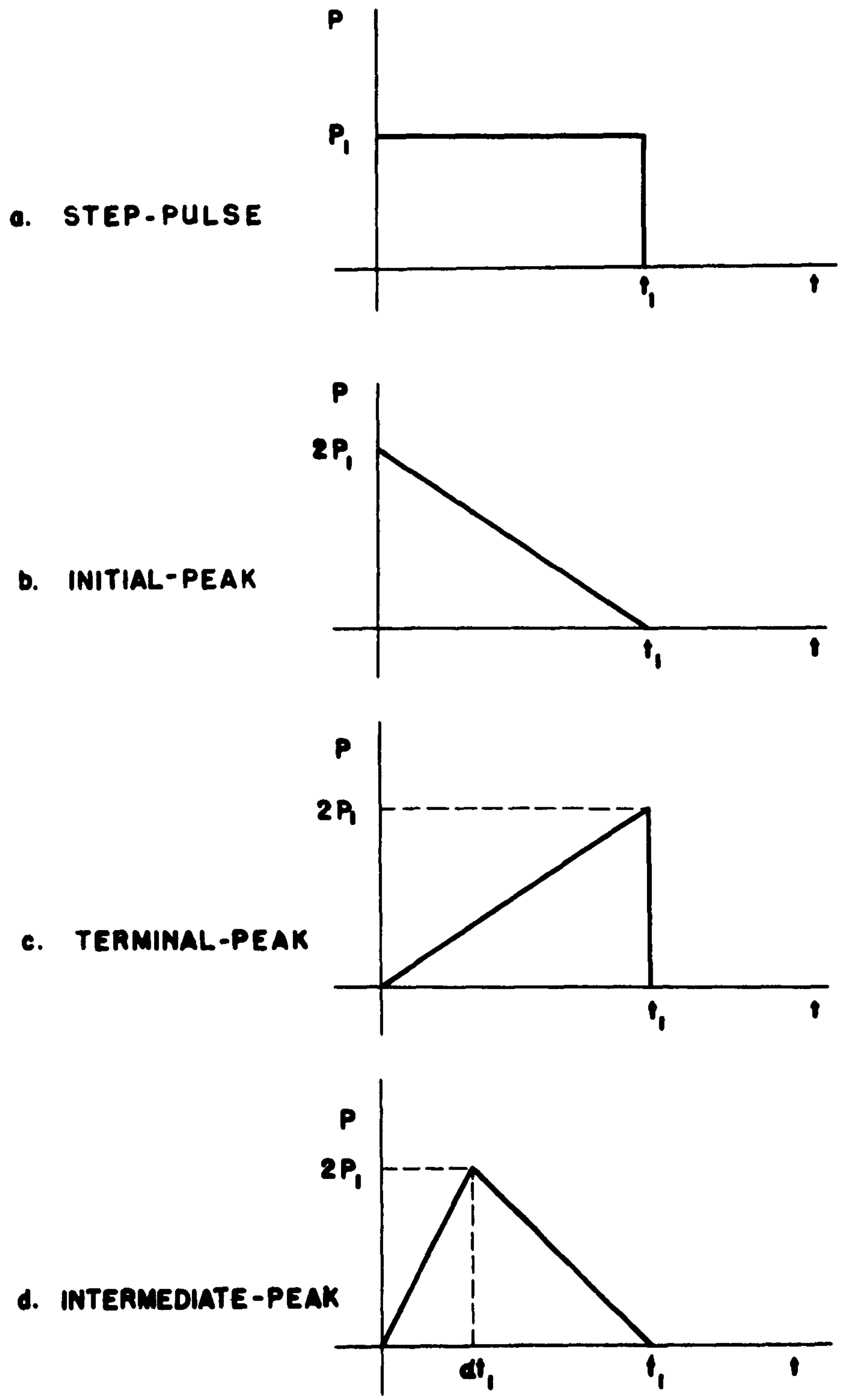

FIG. 3 FORCING FUNCTIONS

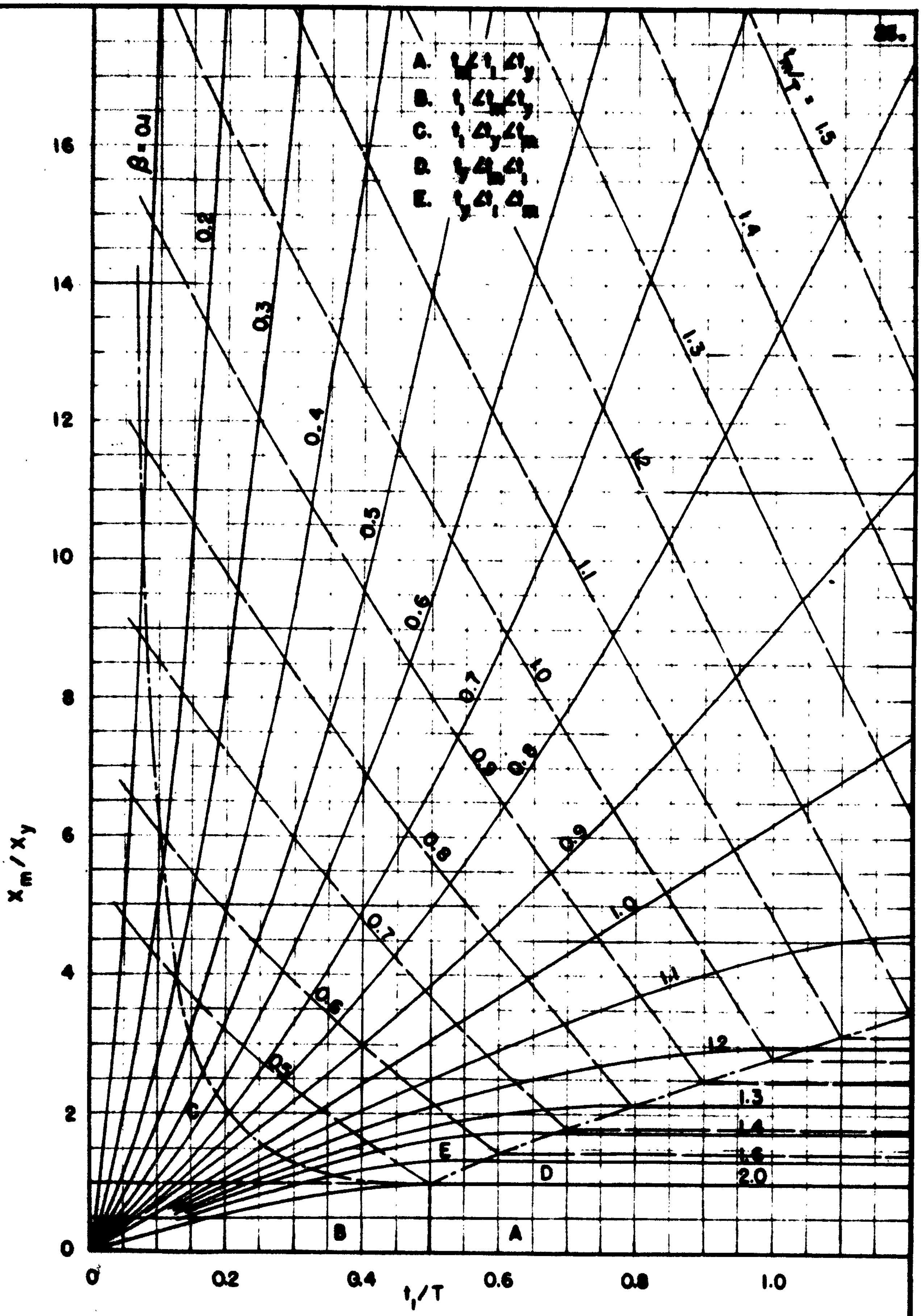

FIG. 4-a MAXIMUM RESPONSE TO STEP-PULSE

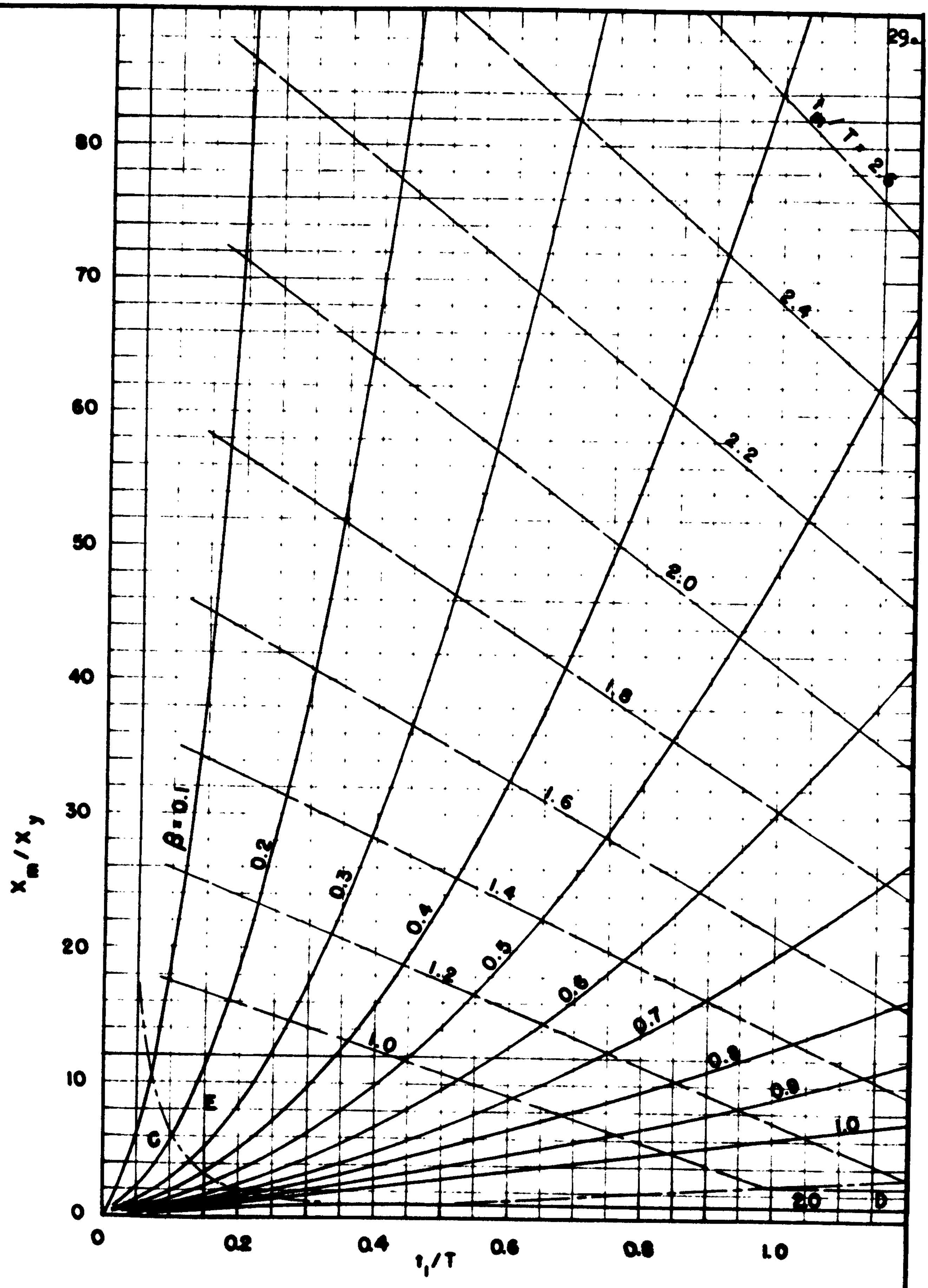

FIG. 4-b MAXIMUM RESPONSE TO STEP-PULSE

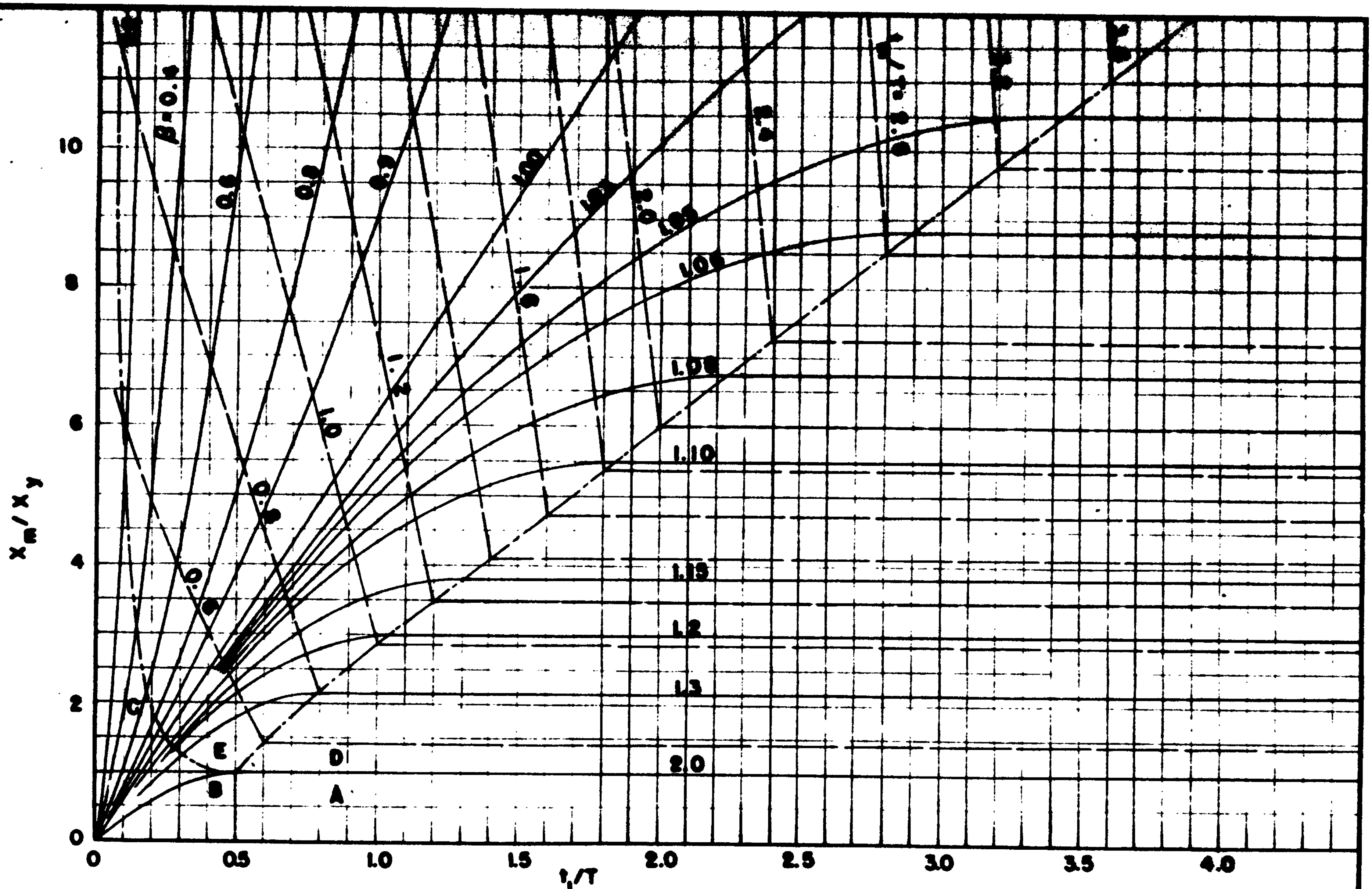

FIG. 4-c MAXIMUM RESPONSE TO STEP-PULSE

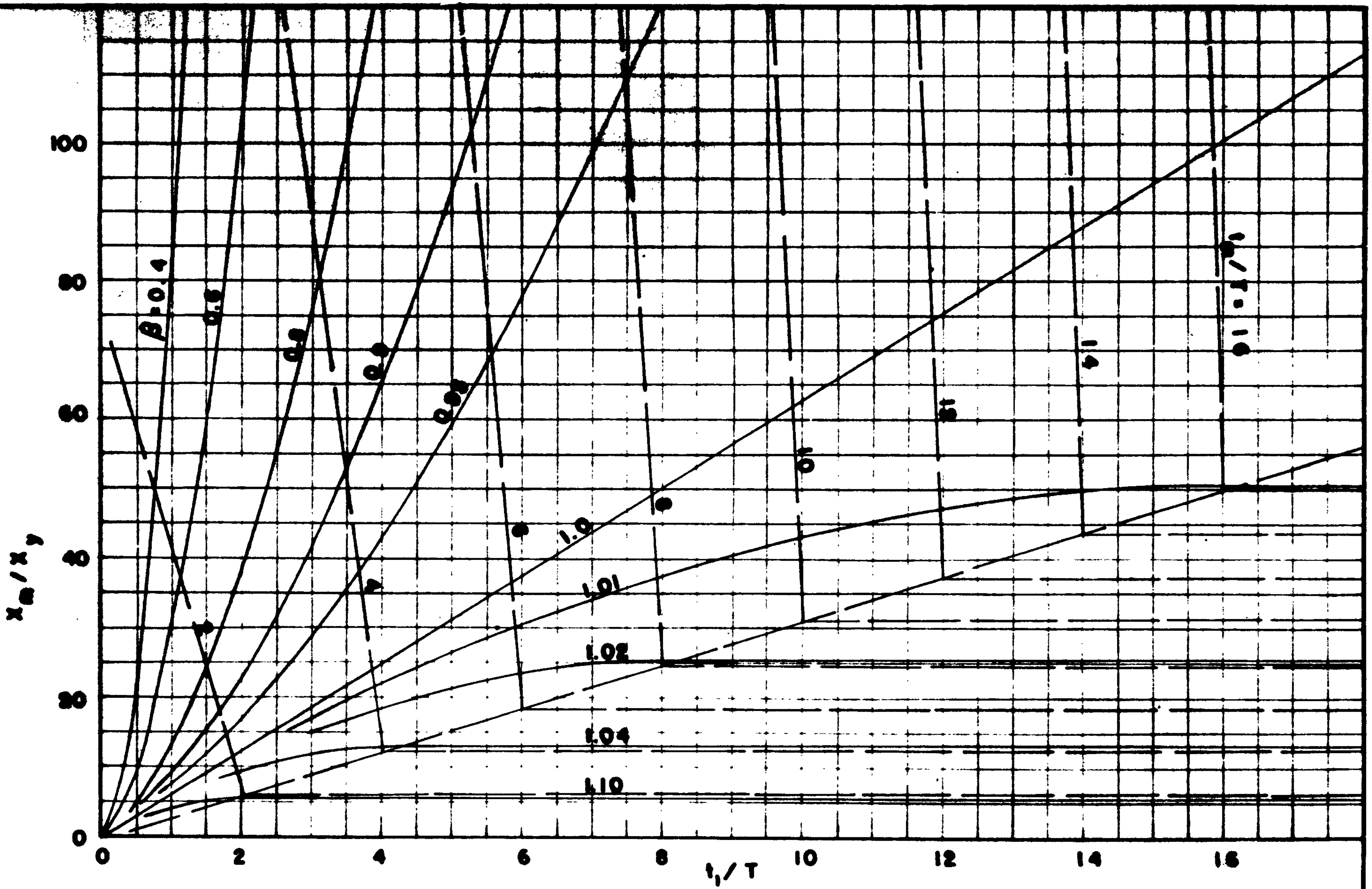

FIG. 4-d MAXIMUM RESPONSE TO STEP-PULSE

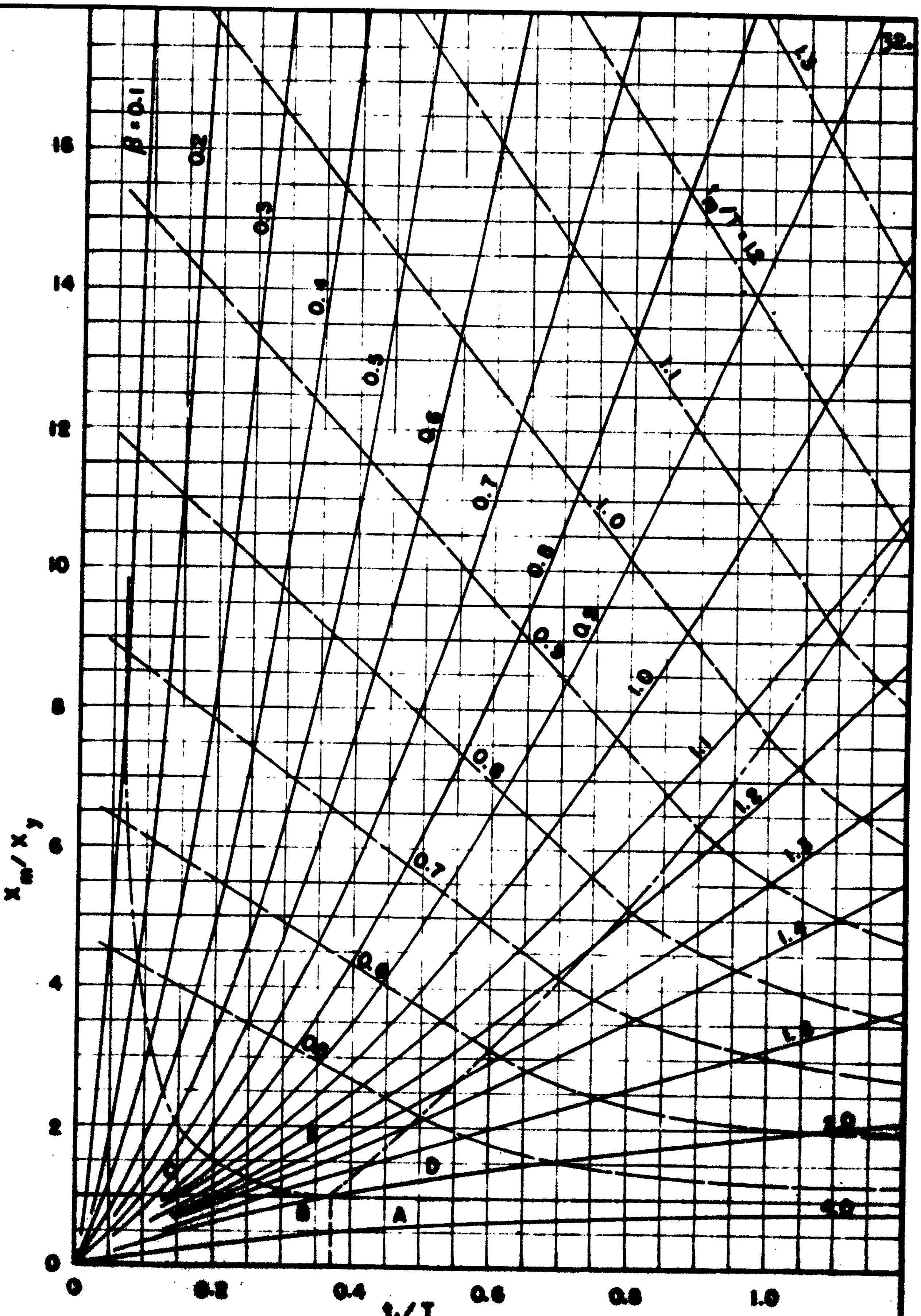

FIG. 5-a MAXIMUM RESPONSE TO INITIAL-PEAK TRIANGULAR PULSE

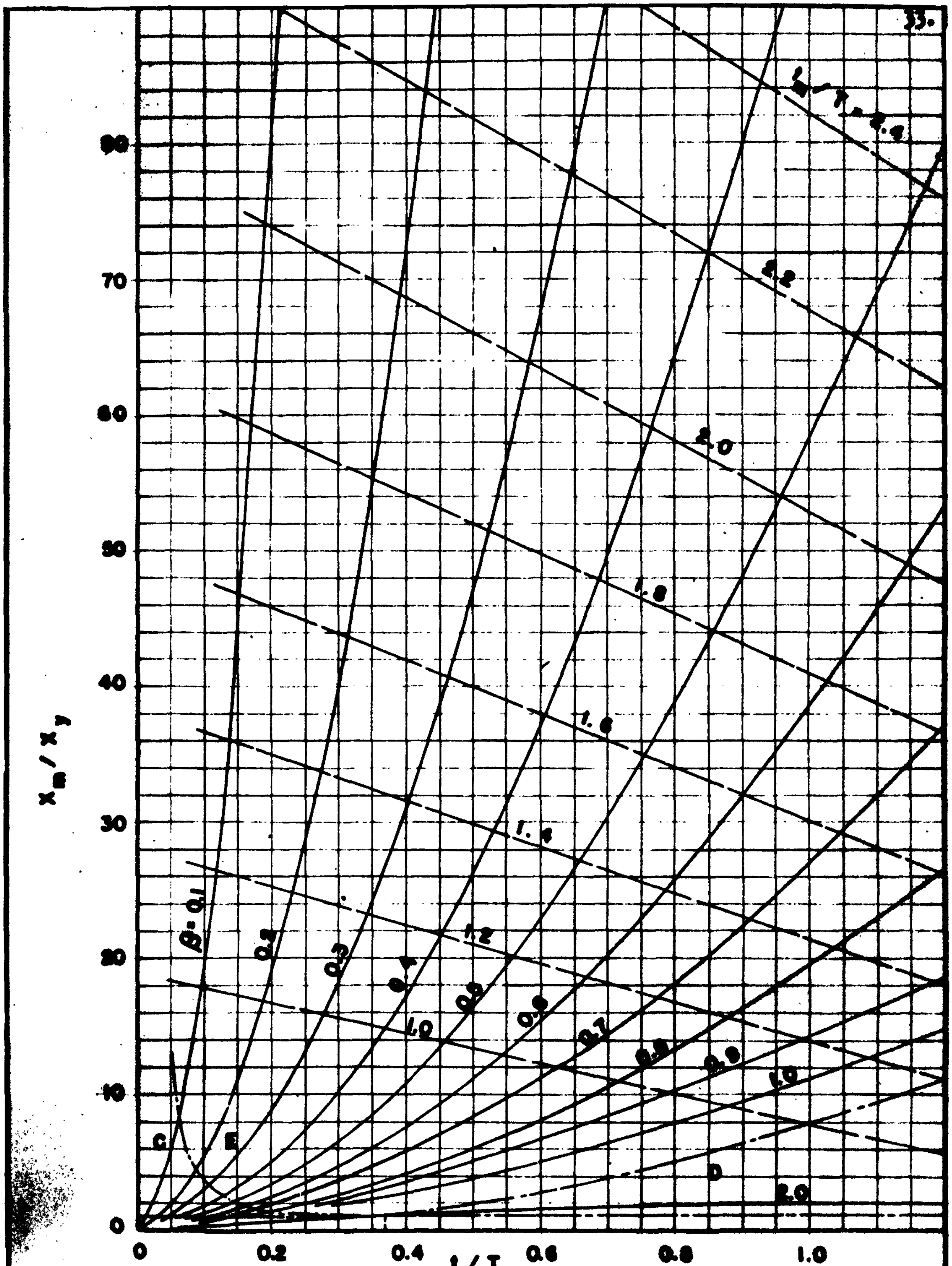

FIG. 5-b MAXIMUM RESPONSE TO INITIAL-PEAK TRIANGULAR PULSE

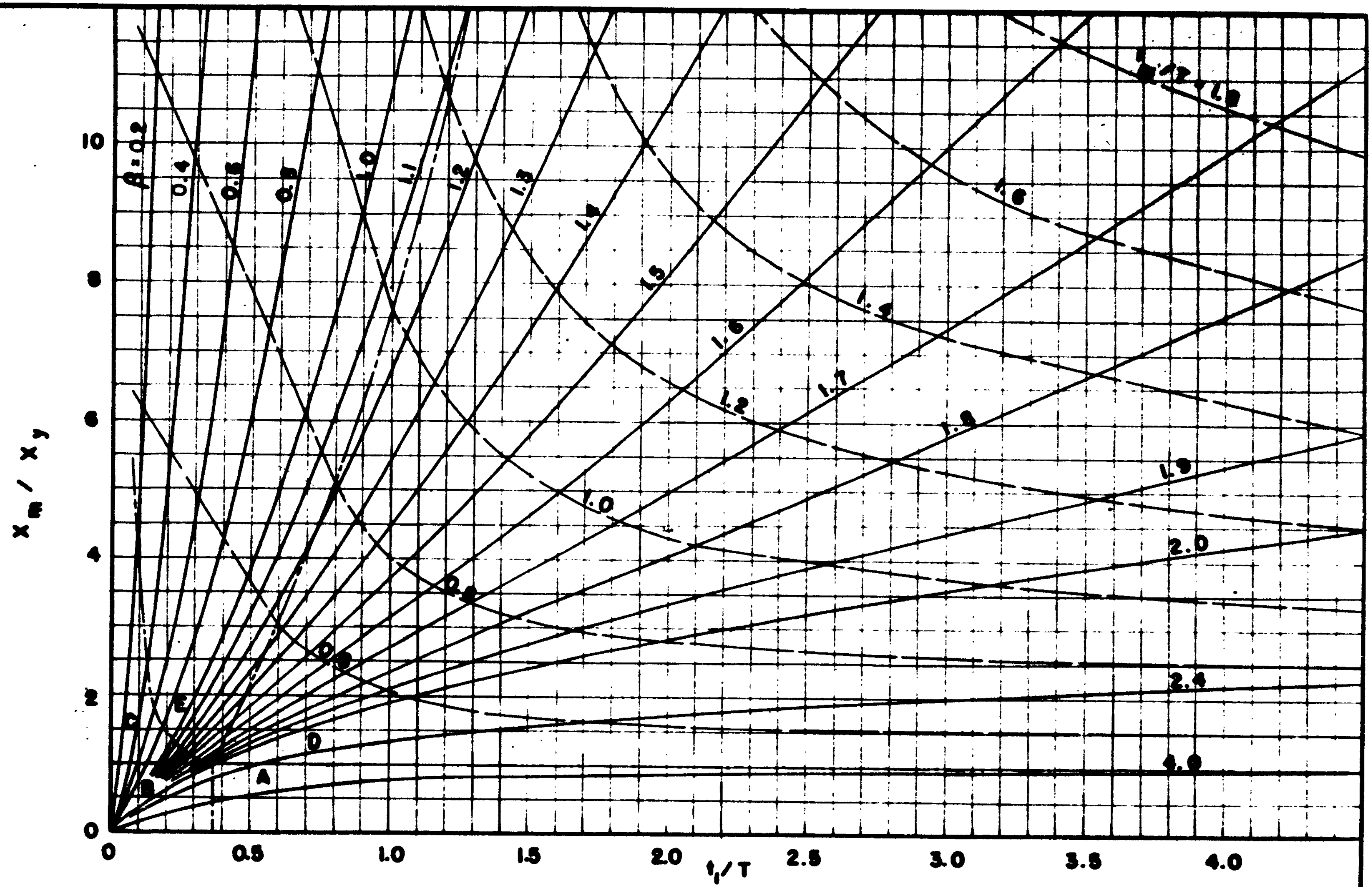

FIG. 5-c MAXIMUM RESPONSE TO INITIAL-PEAK TRIANGULAR PULSE

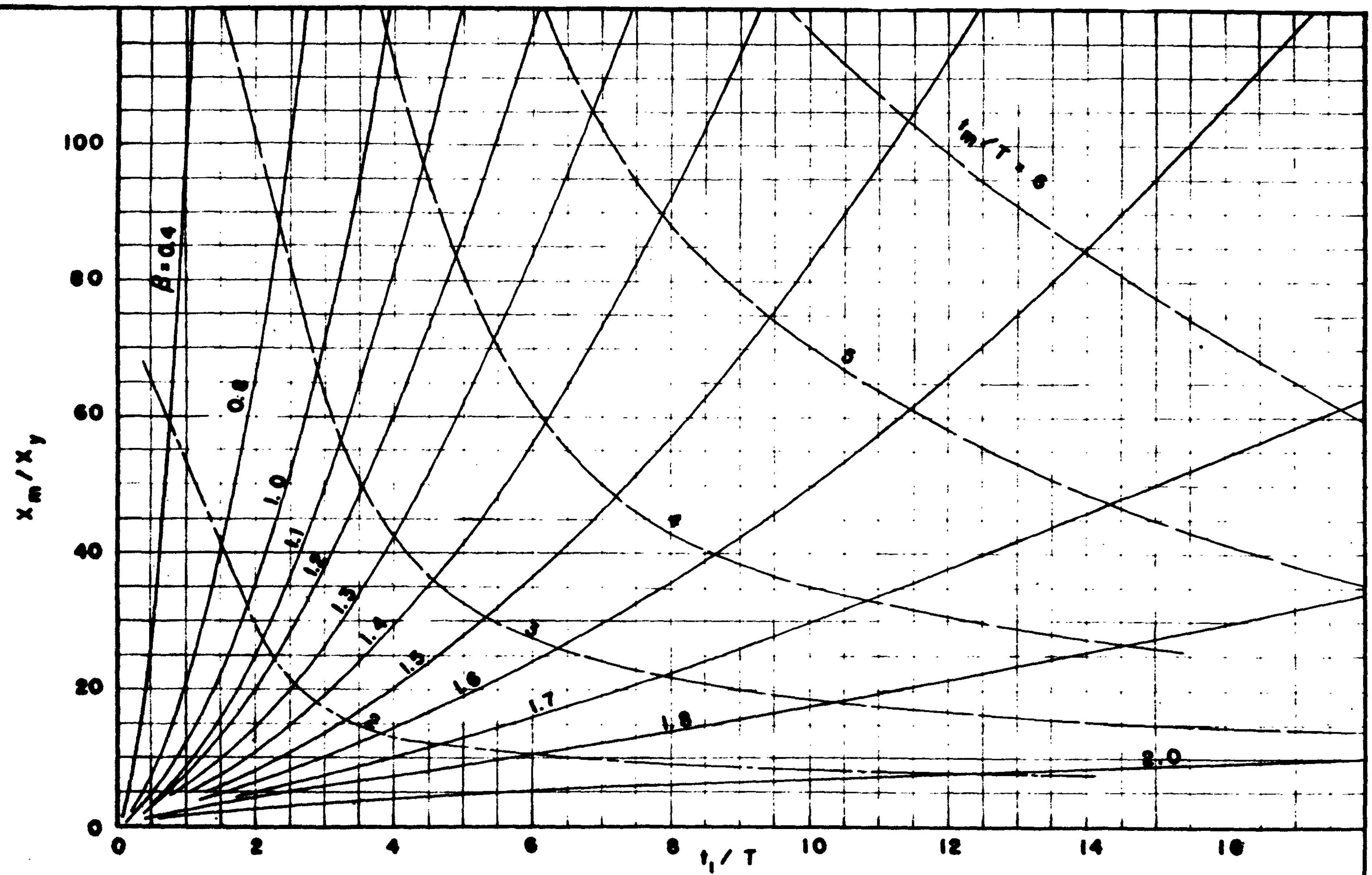

FIG. 5-d MAXIMUM RESPONSE TO INITIAL-PEAK TRIANGULAR PULSE

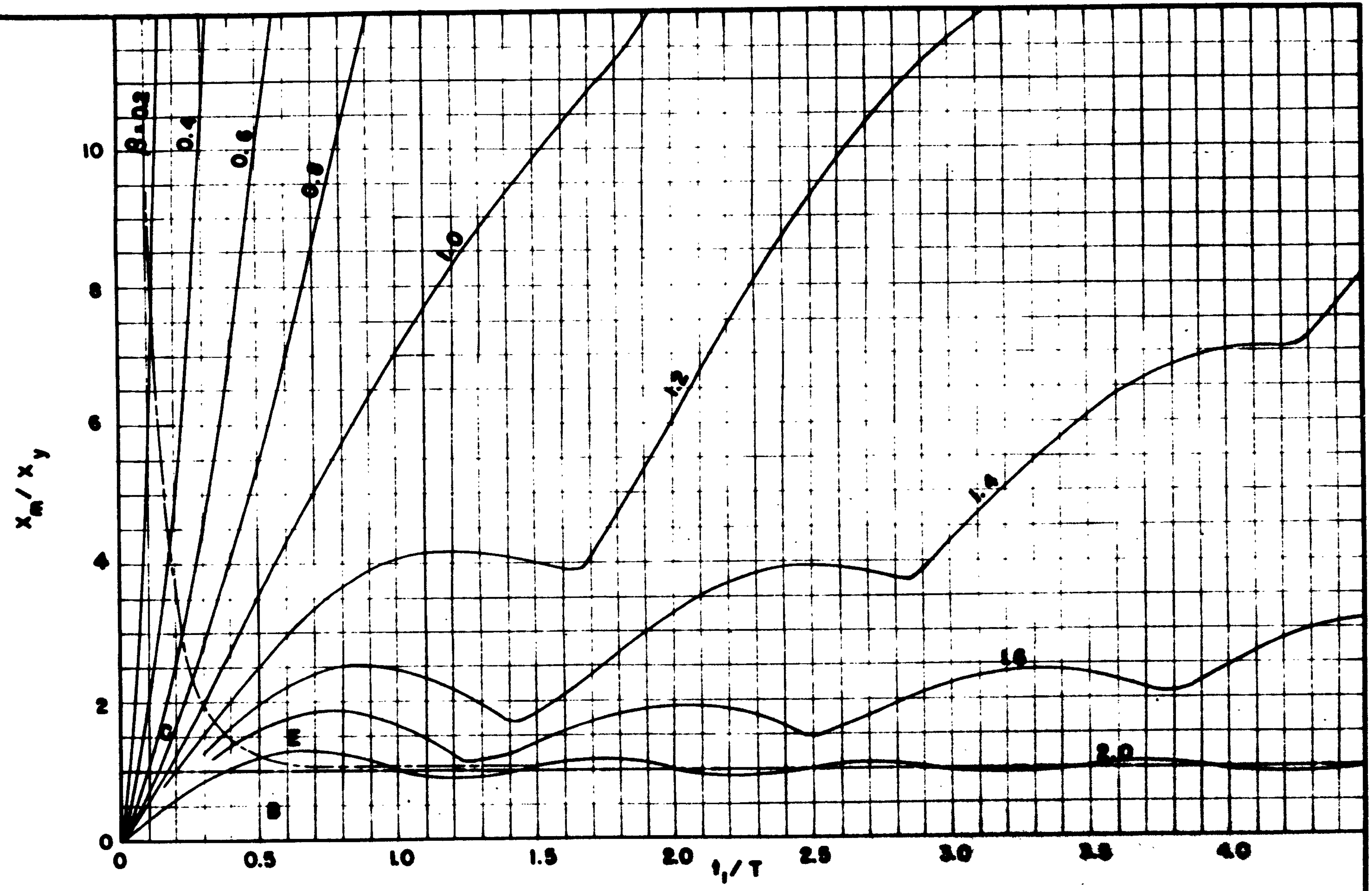

FIG. 6 MAXIMUM RESPONSE TO TERMINAL-PEAK TRIANGULAR PULSE ε=1.0

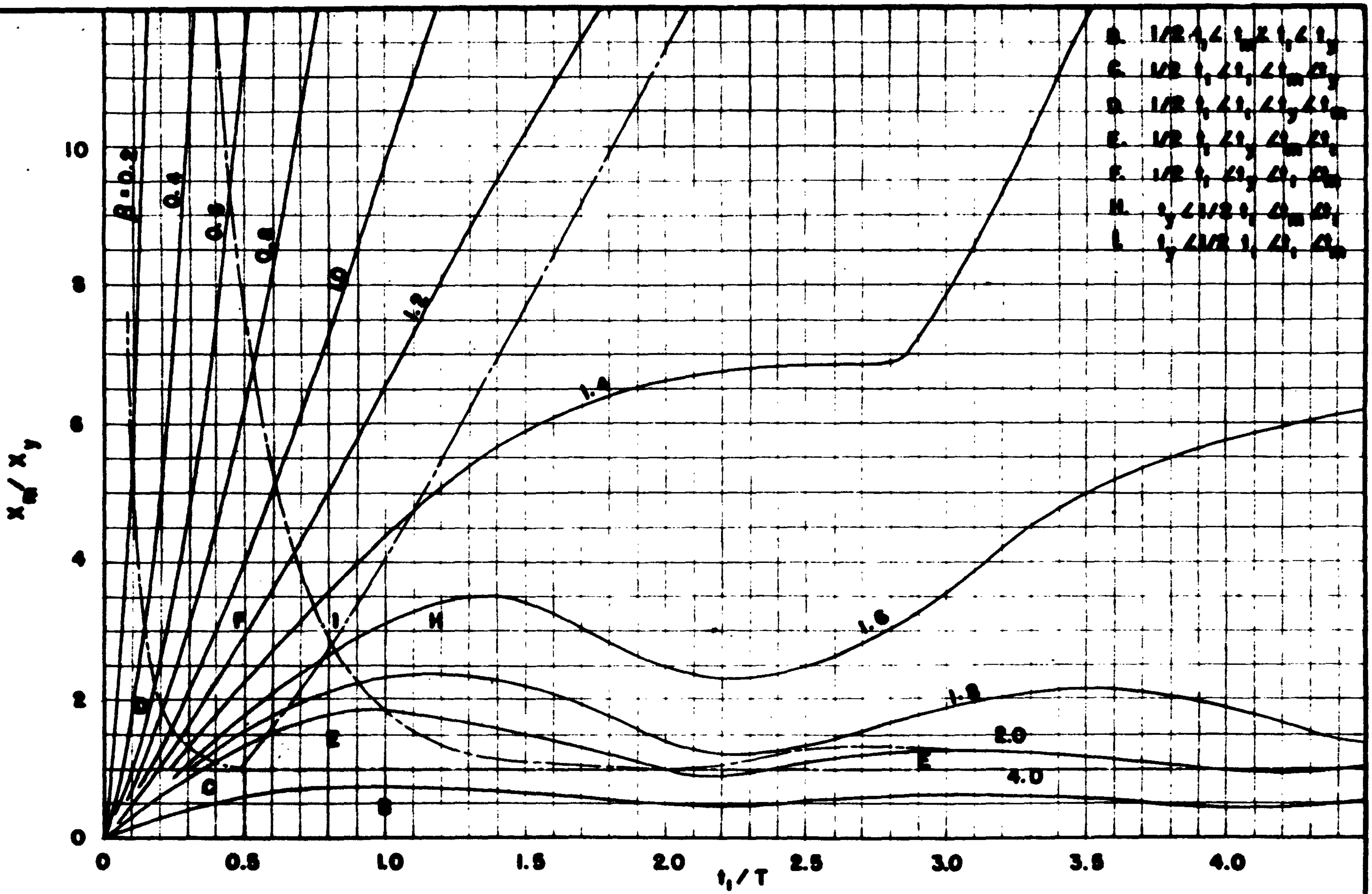

FIG. 7 MAXIMUM RESPONSE TO INTERMEDIATE-PEAK TRIANGULAR PULSE $\alpha = 1/2$

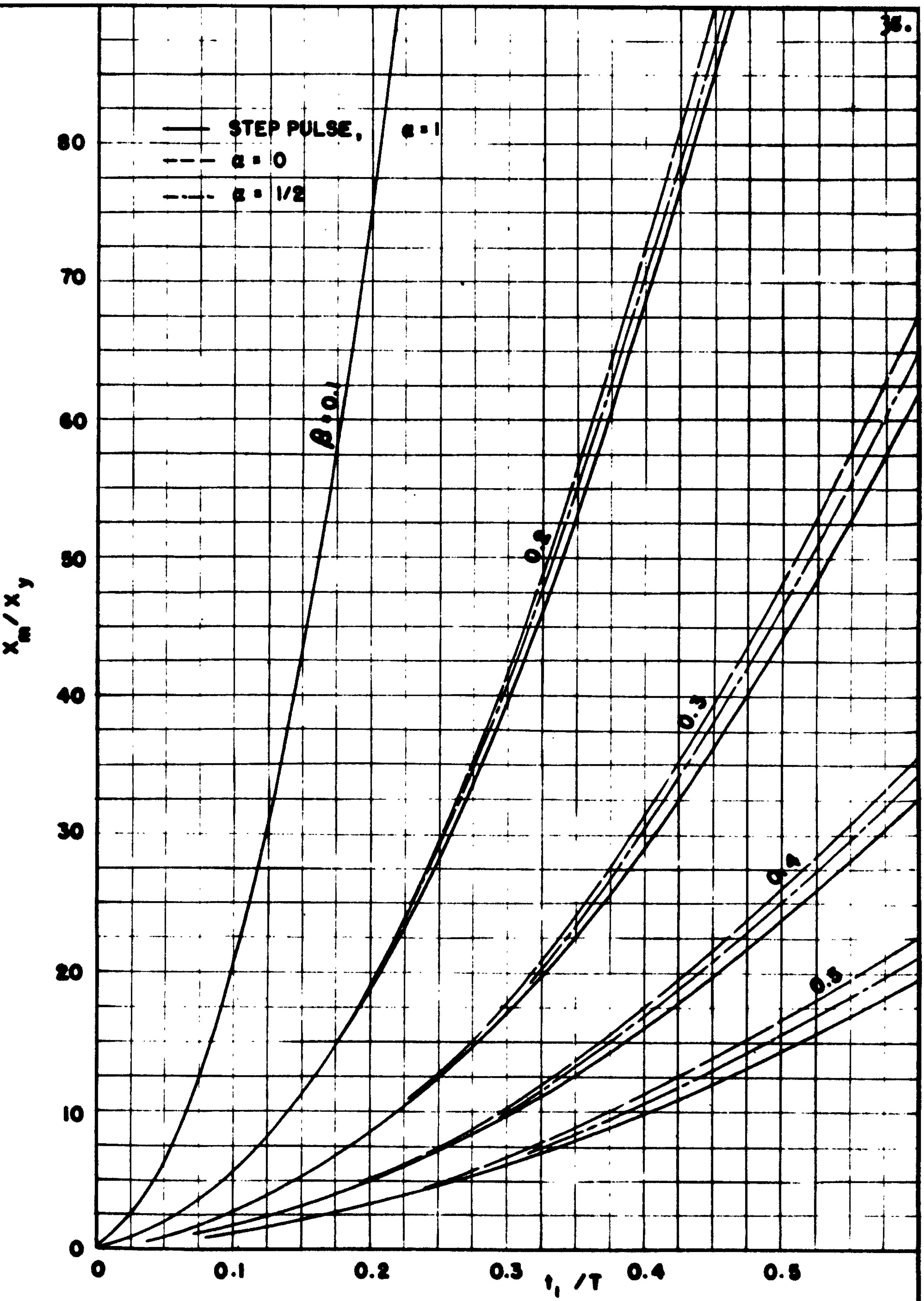

FIG. 8 COMPARISON OF RESPONSE TO DIFFERENT PULSES IN IMPULSE REGION

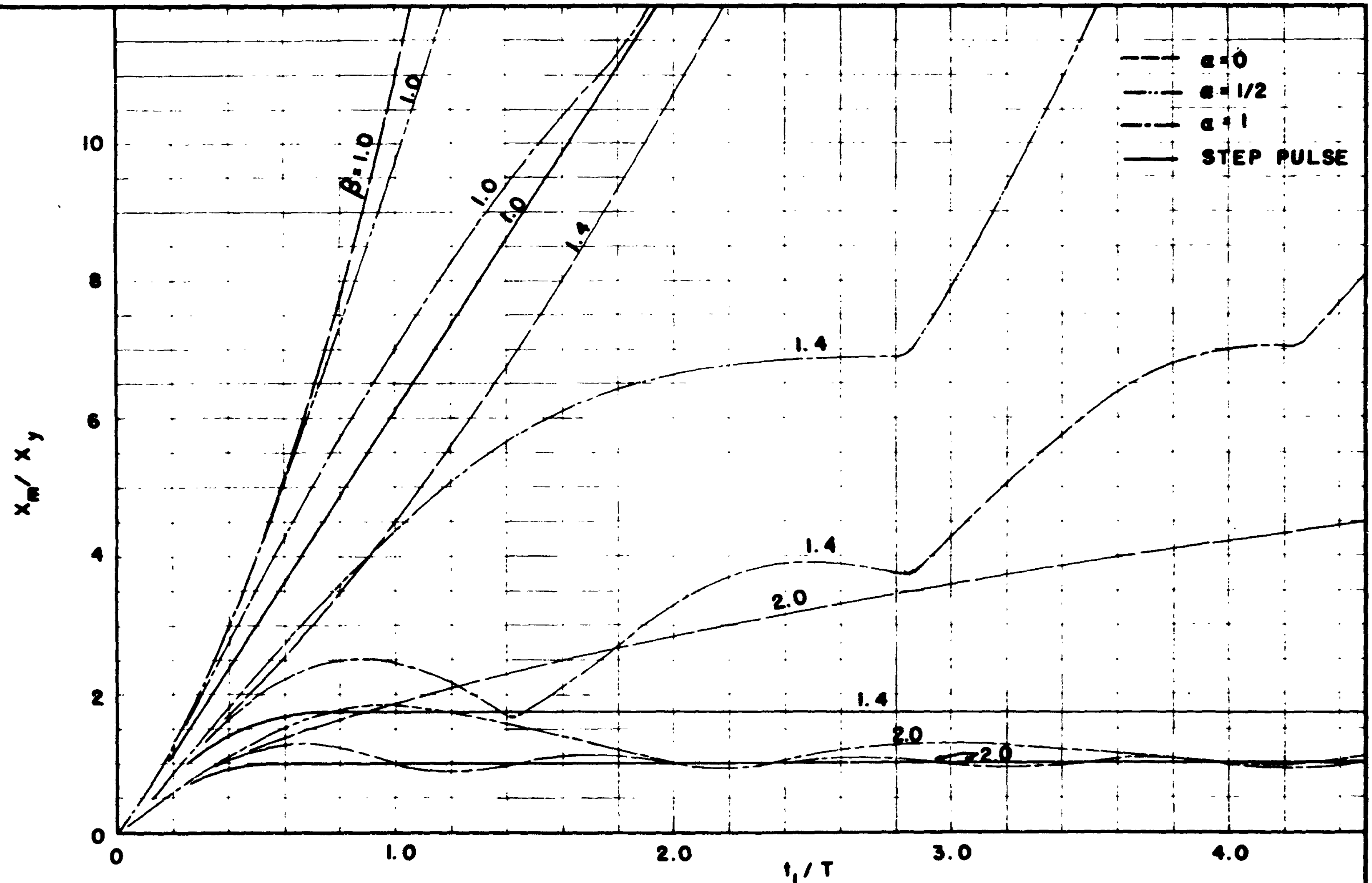

FIG. 9 COMPARISON OF RESPONSE TO DIFFERENT PULSES IN LONG-DURATION REGION

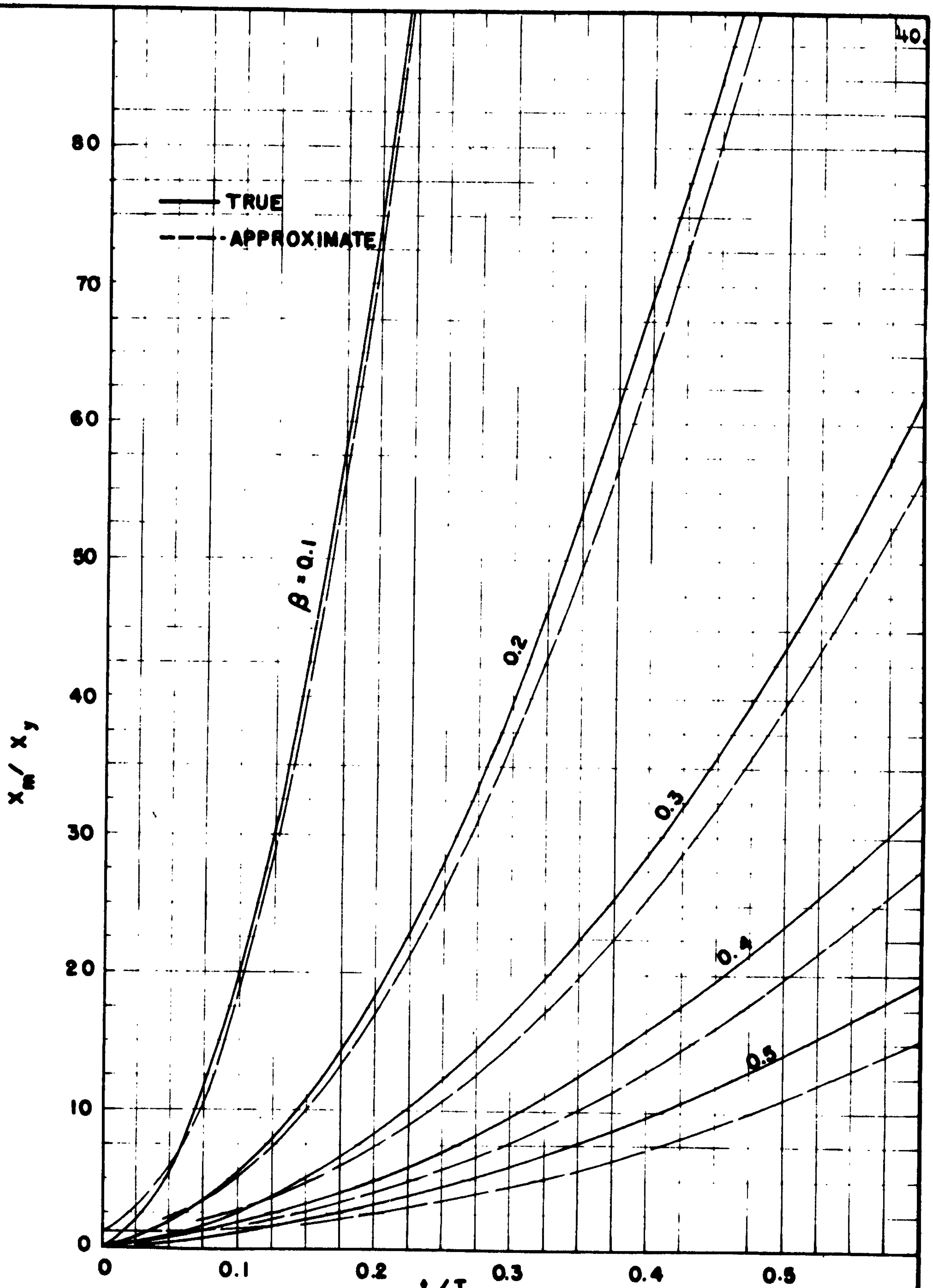

FIG. 10 COMPARISON OF APPROXIMATE AND TRUE RESPONSE TO STEP PULSE IN IMPULSE REGION

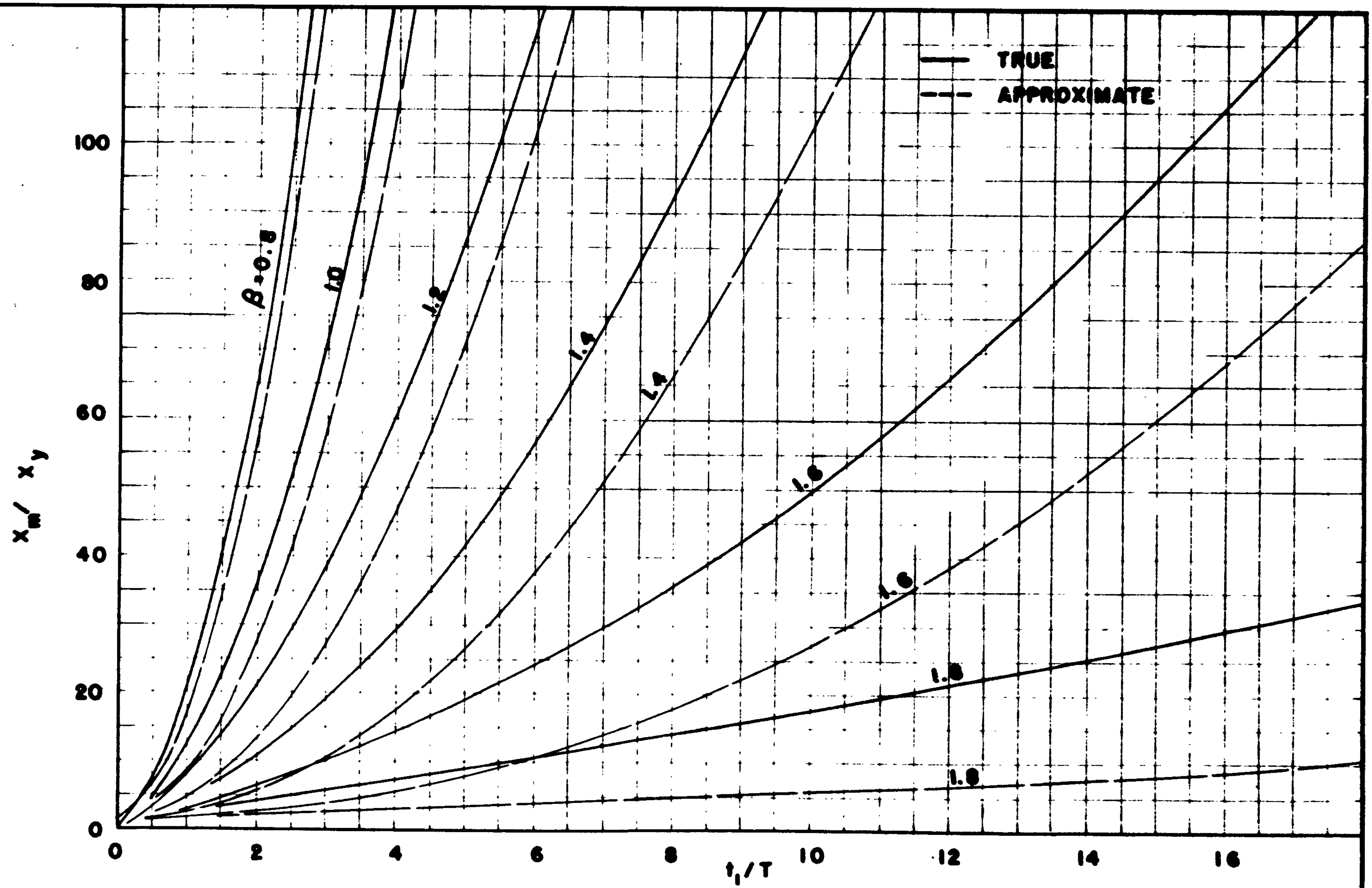

FIG. 11 COMPARISON OF APPROXIMATE AND TRUE RESPONSE TO INITIAL-PEAK TRIANGULAR PULSE IN LONG-DURATION REGION

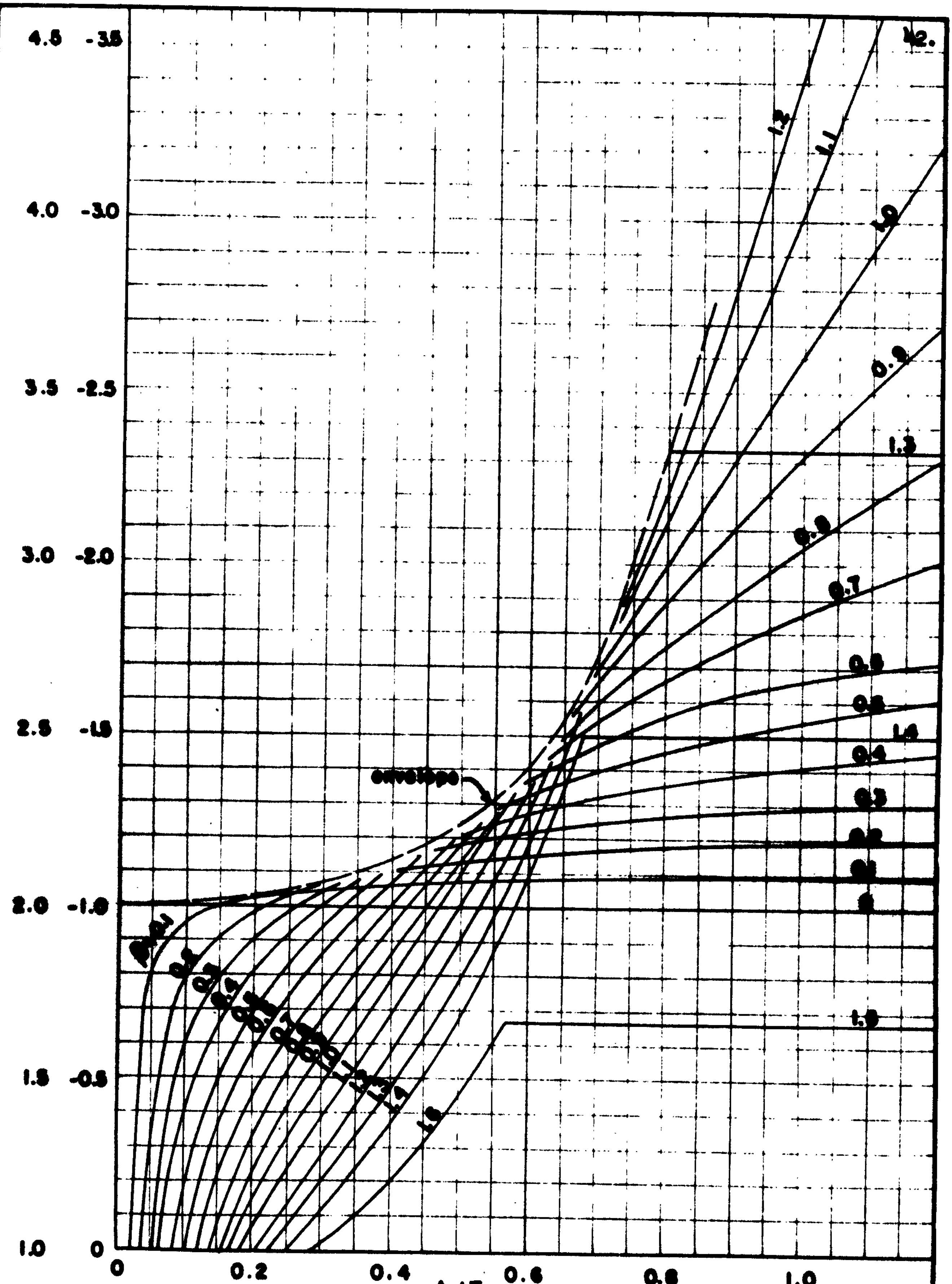

FIG. 12-a AVERAGE LOAD AND YIELD LOAD INFLUENCE FACTORS FOR STEP PULSE

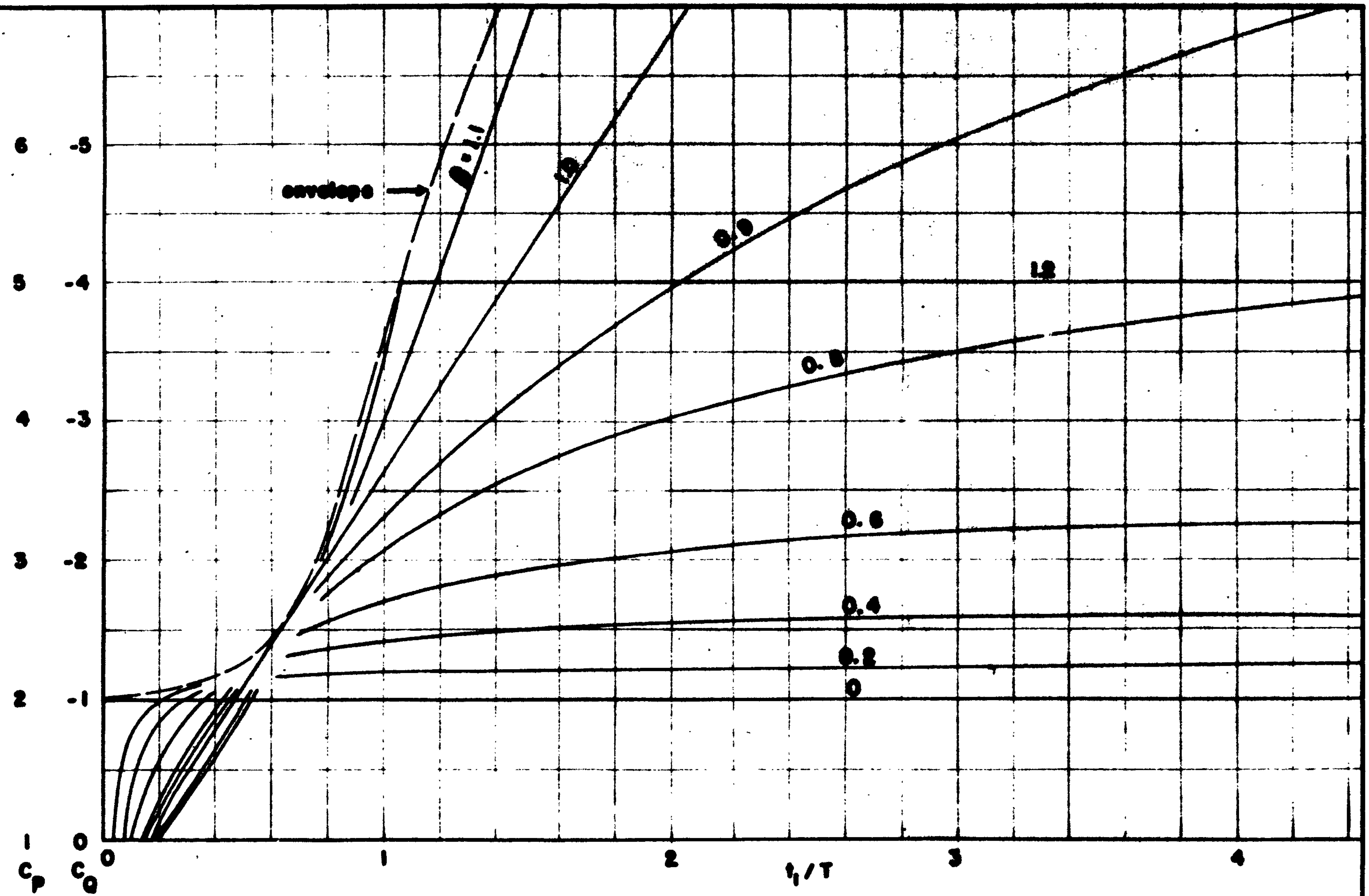

FIG. 12-b AVERAGE LOAD AND YIELD LOAD INFLUENCE FACTORS FOR STEP PULSE

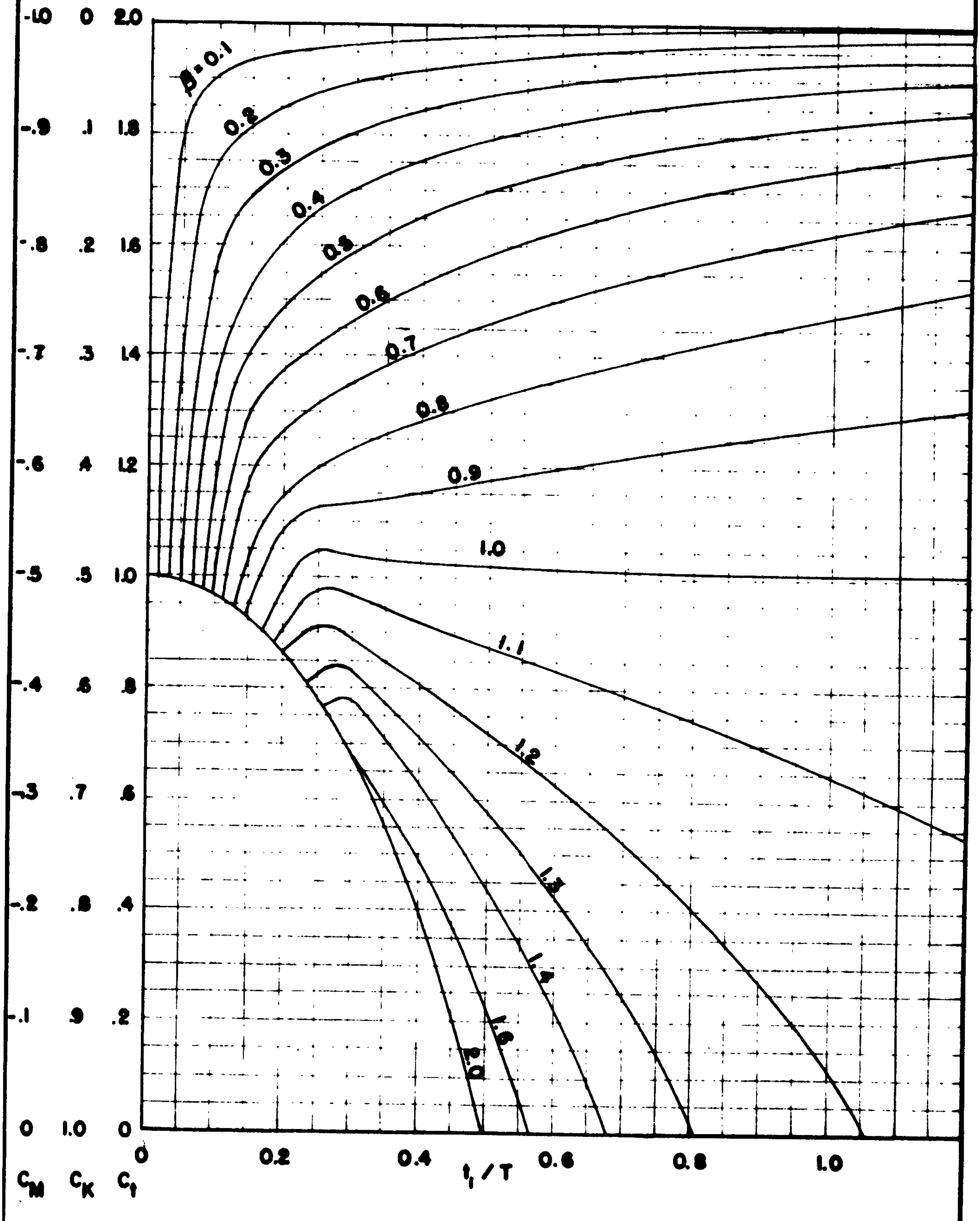

FIG. 13-a MASS, STIFFNESS, AND DURATION INFLUENCE FACTORS FOR STEP PULSE

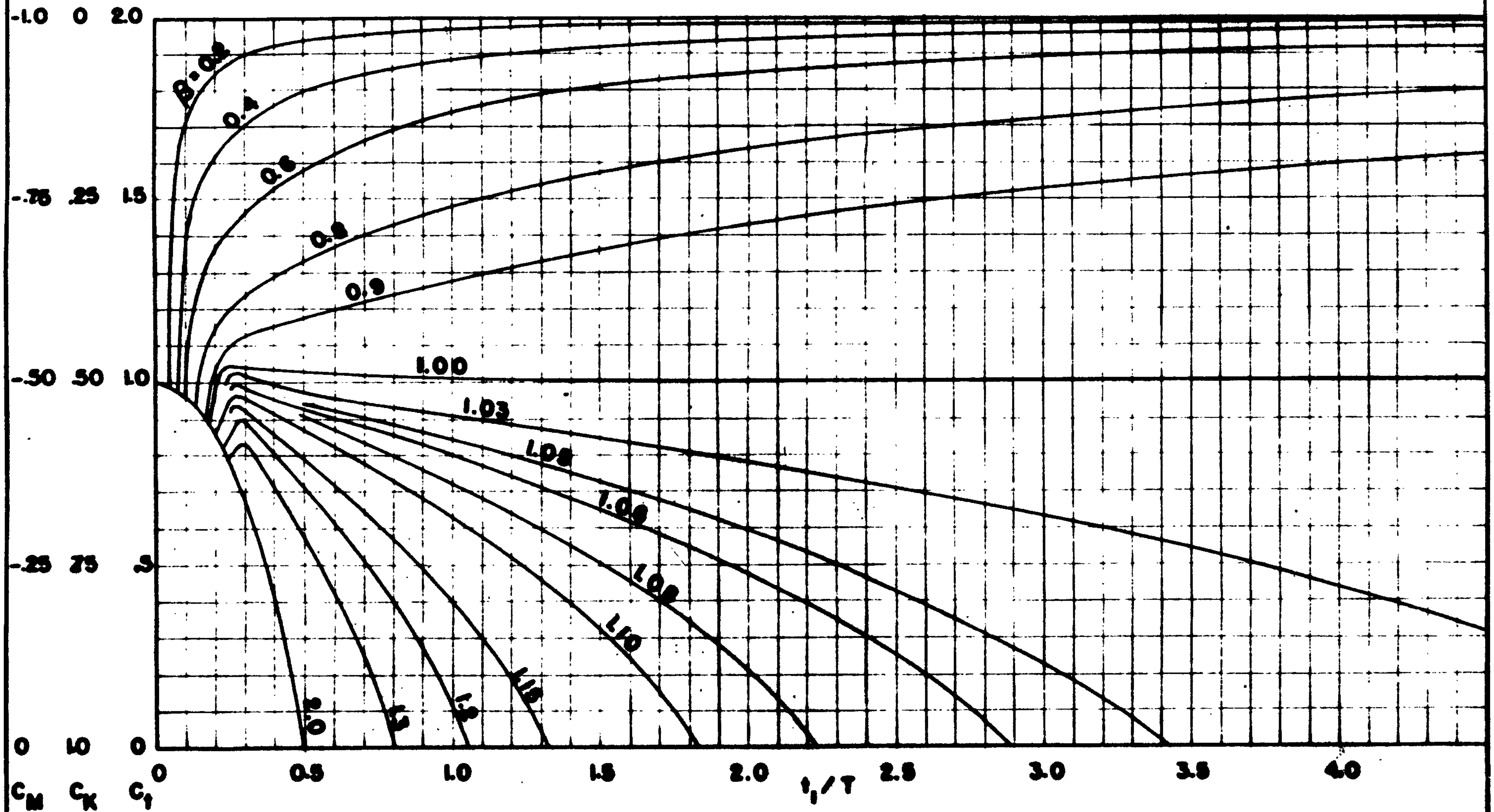

FIG. 13-b MASS, STIFFNESS, AND DURATION INFLUENCE FACTORS FOR STEP PULSE

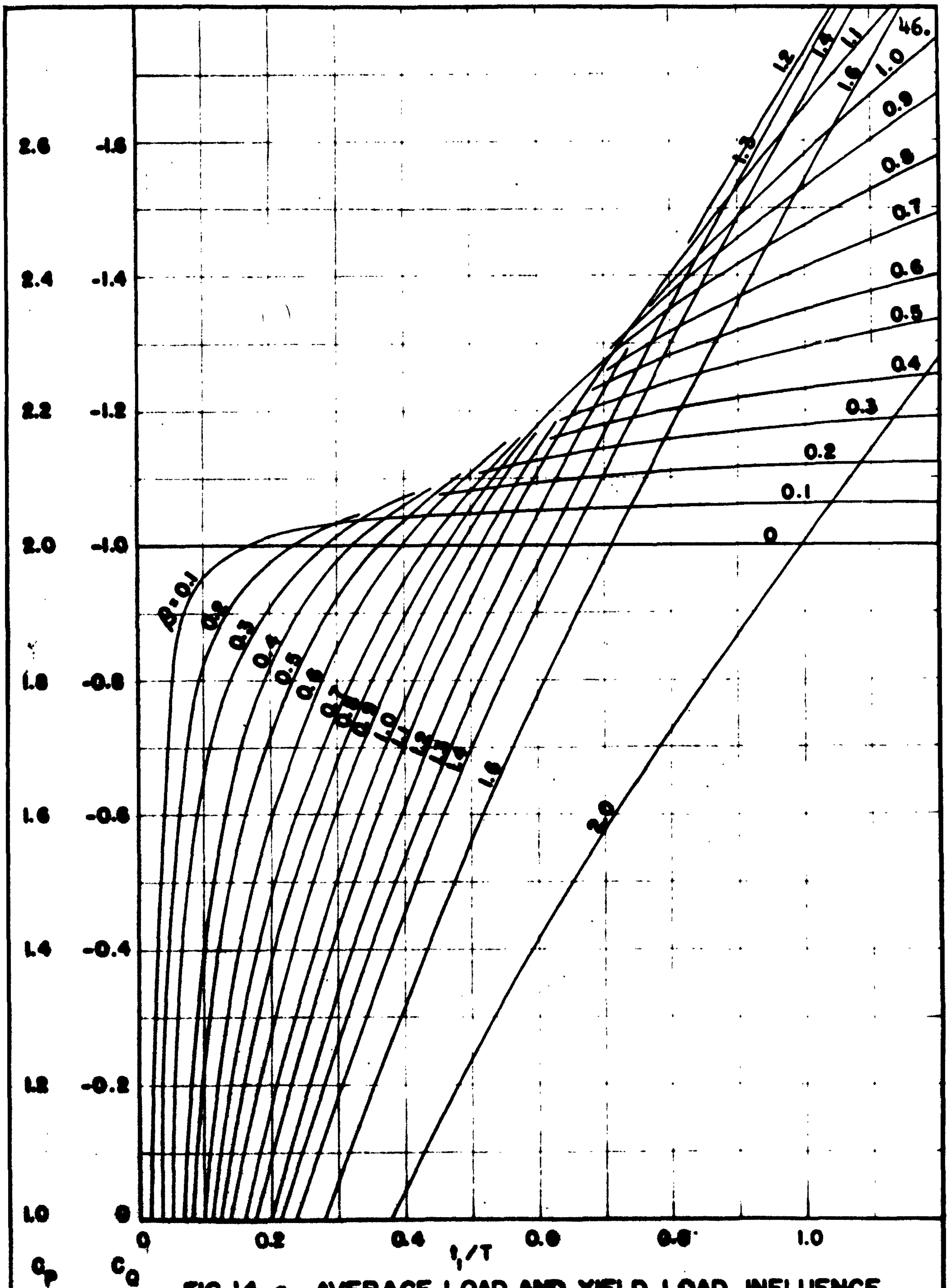

FIG. 14-a AVERAGE LOAD AND YIELD LOAD INFLUENCE FACTORS FOR INITIAL-PEAK TRIANGULAR PULSE

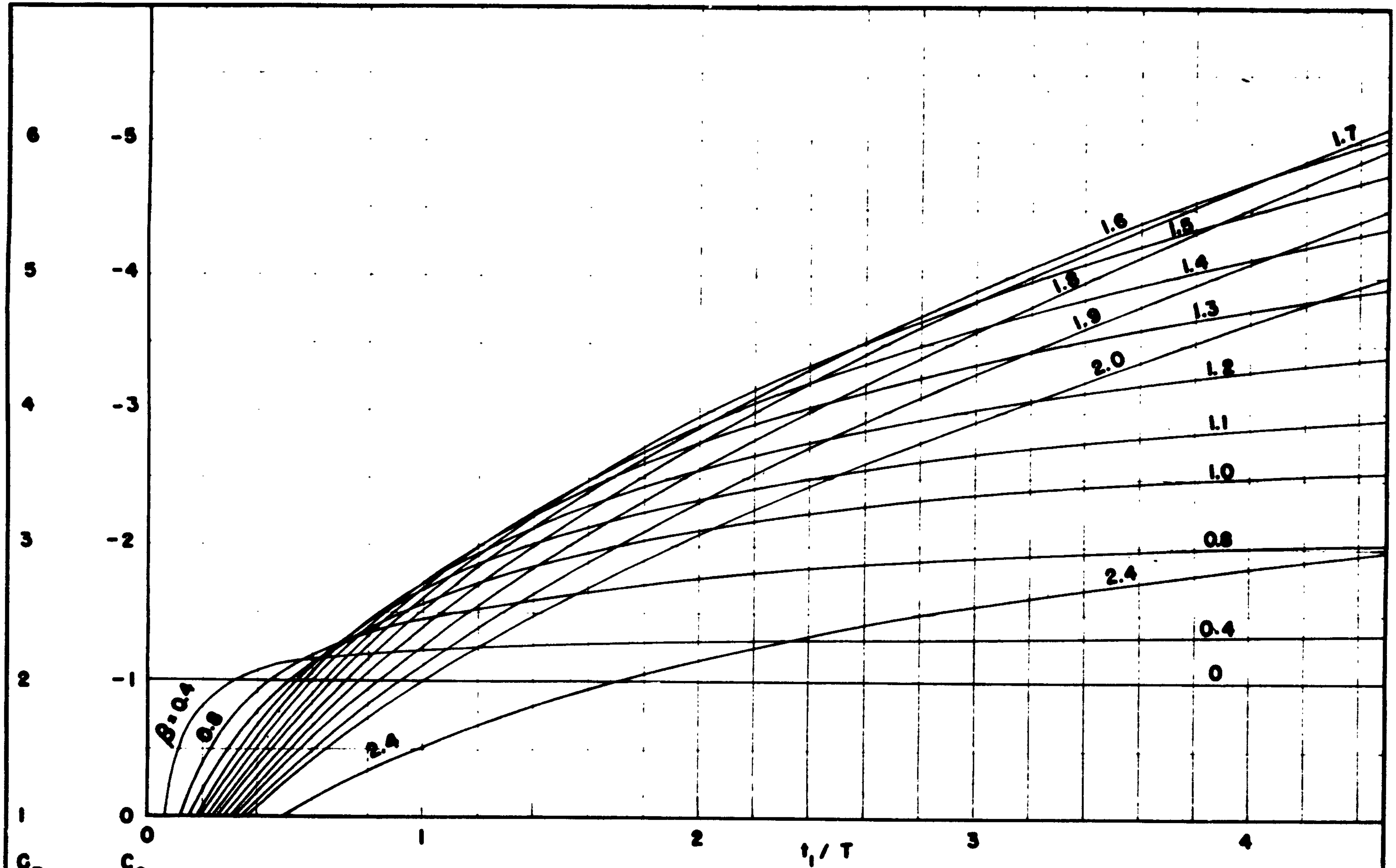

FIG. 14-b AVERAGE LOAD AND YIELD LOAD INFLUENCE FACTORS FOR INITIAL-PEAK TRIANGULAR PULSE

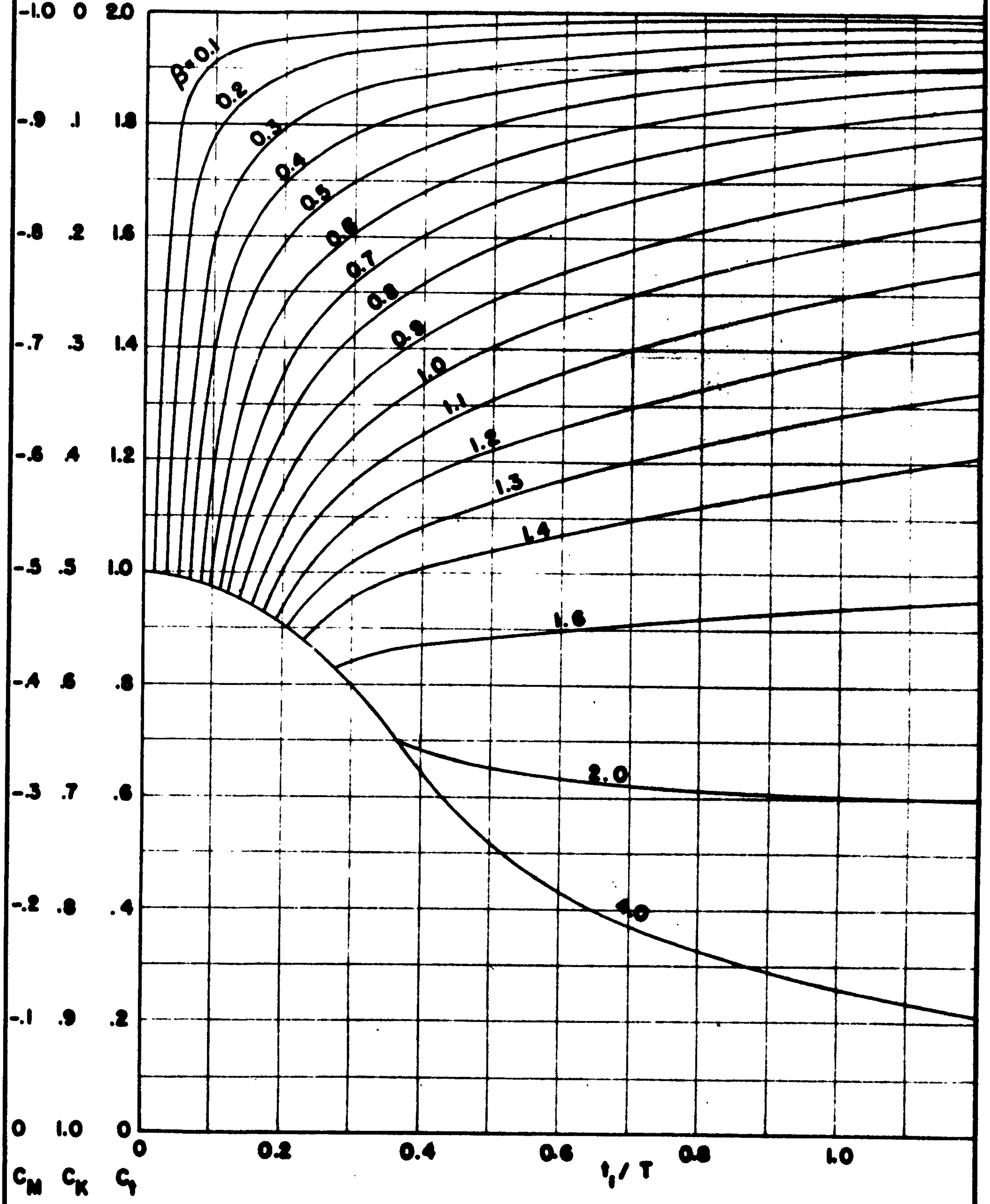

FIG. 15-a MASS, STIFFNESS, AND DURATION INFLUENCE FACTORS FOR INITIAL-PEAK TRIANGULAR PULSE

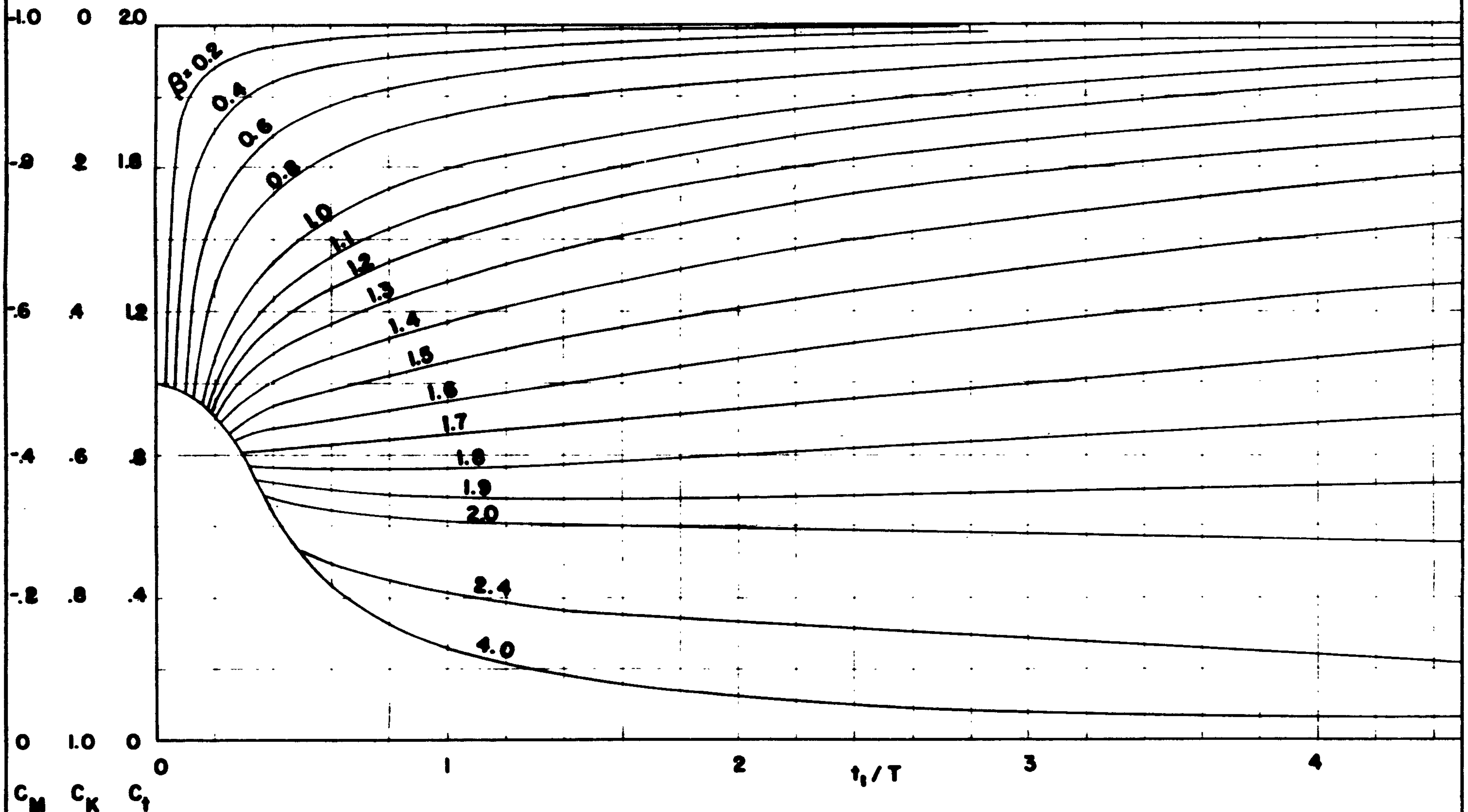

FIG. 15-b MASS, STIFFNESS, AND DURATION INFLUENCE FACTORS FOR INITIAL-PEAK TRIANGULAR PULSE

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